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Some fundamental issues in General Relativity and their resolution

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The purpose of this article is to draw attention to some fundamental issues in General Relativity. It is argued that these deep issues cannot be resolved within the standard approach to general relativity that considers *every* solution of Einstein's field equations to be of relevance to some, hypothetical or not, physical situation. Hence, to resolve the considered problems of the standard approach to general relativity, one must go beyond it. A possible approach, a theory of everything, is outlined in the present article and will be developed in details subsequently.

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I. INTRODUCTION

Historically, Newton based his laws of mechanics on the notion of a particle - a mass-point. Since then, the notion of a mass-point has been an exceptionally useful and important approximation in Physics, in general.

As is well known, in the Newtonian theoretical framework, a physical body is a collection of mass-points. Newtonian laws of motion, together with additional assumptions about the nature of the body, such as its rigidity etc., determine the dynamics of that body under scrutiny. It may be emphasized that these additional assumptions are mainly ad-hoc in nature, but are very useful nonetheless. Further, these ad-hoc assumptions primarily deal with the nature of non-gravitational phenomena affecting the constituent particles of a body. Newtonian inverse-square law of gravitation, Newtonian laws of motion and these additional assumptions then provide the basis for Newton's "mechanical" world-view.

Any radical departure from the Newtonian concept of a point-particle was not evident in Physics in any concrete form until the advent of electromagnetism in its classically complete form. This departure gradually emerged in the field conception of Maxwell and Faraday. With Maxwell and Faraday, the electromagnetic field emerged as a physical entity completely separate in existence from a Newtonian particle. Essentially, a particle, as a conception, has "existence" only at one spatial location while the field, as a conception, has "existence" at more than one, continuous, spatial locations at any instant of time.

However, a point-particle is still needed in the form of a source of the electromagnetic field. In this dualism of a field and a particle, a source particle is necessarily a singularity of the field it generates. For example, an electric charge, as a source, is a *singularity* of *its* electric field. As another example, a material point at rest is represented by its gravitational field which is everywhere finite and regular, except at the position where a material point is located: there the gravitational field also has a *singularity*.

In this dualism, the "gravitational" mass and an electric charge are the physical attributes of a particle as a result of which it *generates* its gravitational and electric fields. In a similar spirit, a particle could be endowed with other physical attributes such as spin, for example.

As a separate remark, it may be also noted here that, within the field conception, the Newtonian concept of instantaneous action at a distance gets, naturally, replaced with that of an interaction mediated by the field carrying the disturbance to a different location.

Now, let us, for the moment, assume that space and time do not play any "dynamic" role in physical phenomena but provide only an inert stage for them. This, for example, has been the situation with all the pre-general relativistic theories of the physical phenomena.

Then, a physical body is imagined to be a collection of particles acting as sources (or singularities) of different fields depending on the attributes chosen for the particles making up the body.

Now, we could approximate any collection of mass-points by average mass-density. This is permissible since a volume form, in Cartan's sense, of the background space is available to define a smooth distribution of sources. (This will be commented on at a later stage.)

This is the *fluid description* of sources. As is well-known, a continuous distribution allows fields to be defined at points outside as well as "inside" the distribution of sources itself since the field singularities have been done away with.

Now, since the volume-form of the space is well-defined at all points of the background space, well-posed problems could be formulated for which some fluid distribution of sources generated a "smooth" field even when the field is singular at the locations of its sources.

Similarly, some smooth distribution of sources could also be replaced by a "concentrated" source, a very useful approximation in Physics. For example, a charge distribution could be replaced by a point charge or a mass distribution could be replaced by a point mass.

In either case, it can be uniquely ascertained as to when the fluid description fails for a given situation. The inert stage of space and time provides the means of these assertions. Essentially, the fluid description fails when the volume of interest is so small that fluctuations in variables due to "individual" particles become significant.

II. SITUATION IN GENERAL RELATIVITY

General Theory of Relativity (GTR) changes the canvas of this painting drastically. A spacetime becomes a "dynamic" stage for physical phenomena. The stage changes with the events and events in turn change as the stage changes.

Firstly, a point-particle in GTR is, necessarily, a *curvature singularity* of the spacetime.

A point-particle has existence only at one spatial location and everywhere else, except at this *special* location, it is the non-singular gravitational field of the particle that "exists". In its spherically symmetric spacetime *geometry*, all the points, except for the location of the mass-point, are then "matter-wise equivalent" to each other as the rest of the manifold describes the vacuum "gravitational field" that only diverges at the location of the mass-point unless, perhaps, we include the self-field in some way. This *geometry* then should not "include" the location of the mass-point. Then, any mass-point is a curvature singularity of the manifold describing its "exterior" gravitational field.

Addition of other attributes of a particle, like charge, spin etc., does not change this situation. The Kerr-Newman family exists in GTR and the point-mass with charge and spin is a spacetime singularity in it. (The issue of the inadequacy of such a description of a physical particle would have arisen if we had no Kerr-Newman family of spacetimes in GTR. We could then have said that the pure mass-point is an inadequate description of a "physical particle" and the addition of other attributes "takes us away from General Relativity" to some new theory with some new description of a physical particle.)

If we base our picture of the physical world on the notion of a point-particle then, a physical body will be a collection of point-particles, a collection of spacetime singularities. As was the case with Newton's mechanical world-view, we will have to *assume* that non-gravitational interactions of these point-particles will be responsible for the stability, rigidity etc. of physical bodies.

Now, if there existed other particles then, the spacetime geometry would be *different* from that of a single point-particle and the geodesics of that geometry would not be those of the geometry of a single point-particle.

Furthermore, non-gravitational interactions between various particles will have to be incorporated in the geometry, as attributes of a particle. This is also a relevant issue.

Strictly speaking, a path of a point-particle is not a geodesic since the mathematical structure defining a geodesic breaks down at every point along the path of the singularity.

We may surround the singular trajectory of a point-particle by an appropriately "small" world-tube, remove the singular trajectory and call this tube the "geodesic" of the particle. But, the new spacetime is *not* the original spacetime and, by the physical basis of GTR, it corresponds to different gravitational source. Therefore, this procedure is not at all satisfactory.

We may, along a geodesic, change the spatial coordinate label(s), by which we "identify the singularity", with respect to the time label of the spacetime geometry and may term this change the "motion" of a particle along a geodesic. However, this is only some "special" simultaneous relabelling of the coordinates. But, a particle in the spacetime does not "move" at all.

The motion of a particle must be a "singular trajectory" in the spacetime if a particle, the singularity, "moves" at all. It is also equally clear that any singular trajectory is *not* a geodesic in the spacetime since such a trajectory is *not* the part of the smooth spacetime geometry.

Then, strictly speaking, geodesics of a spacetime do *not* provide the *law of motion* of particles. Thus, strictly speaking, we have to specify *separately* the law of motion for the particles. And, GTR does not, strictly speaking, specify this law of motion for the particles. This peculiar situation is quite similar to that of the Newtonian theoretical framework then.

Ignoring these difficulties, we may now replace a collection of point-particles by that of their smooth distribution. But, a new difficulty, not encountered

in Newton's theory, arises as particles are spacetime singularities.

Essentially, a spacetime geometry with "curvature singularities" is being replaced by a smooth geometry using the volume-form of the geometry with curvature singularities. Therefore, such a construction has its own pitfalls.

A point-particle is an essential, non-removable, singularity of the spacetime geometry. Thus, approximating a geometry with a singularity by any smooth manifold means that an entirely different source of gravity is acting in the latter situation. This is, physically, not acceptable.

Also, replacing a collection of point-particles by a smooth manifold is mathematically unacceptable for the following reasons.

Let us attach a "weight function" to each mass-point in our collection. Then, the weight-function must "diverge at suitable rate" at the mass-point to balance the field-singularity at its location. The resulting "weighted-sum" should provide a smooth "volume-measure" as well as a smooth density with respect to the volume-measure of the geometry (that is being replaced).

In terms of the differentials of measures, $d\mu = \rho(x)\, d\mu_o$ where μ is the volume-measure of the smooth geometry, μ_o that of the geometry with singularities and $\rho(x)$ is the required density. Any smooth $\rho(x)$ is *impossible* if the geometry of μ_o has essential curvature singularities. To keep μ smooth the density $\rho(x)$ must diverge at the curvature-singularities of μ_o.

In the newtonian situation, the geometry of μ_o-measure and the geometry of μ-measure, both, are well-behaved. It is *not* the background spacetime which is singular but only some field defined over it that is singular. If we now select some appropriate smooth $\rho(x)$, in the above construction, it can "replace the singularities" of the field by means of suitable weight-functions correcting the field-singularity at locations of mass-points. Therefore, there are no fundamental difficulties of any kind with this procedure in the newtonian theory that deals only with fields defined over a smooth, flat, spacetime manifold.

Moreover, consider that a smooth spherical spacetime has "mass function" $m = 4\pi \int \rho r^2 dr$ where $\rho(r)$ is the *smooth* energy density that is non-vanishing in some region around $r = 0$ and r is suitably defined radial coordinate. As we approach the point $r = 0$, $m \to 0$ since ρ is a non-vanishing function of r in that region.

But, the fluid approximation of sources must fail as the "volume of interest" is so small that fluctuations in density due to "individual" particles become significant much before we approach the point $r = 0$. Then, in a spherically symmetric distribution, there is a mass-point exactly at $r = 0$ or there is none. Both these possibilities are certainly allowed, even generally. Then, $m \to 0$ or $m \nrightarrow 0$ at a location inside the fluid.

In the newtonian case, we could avoid these problems by appealing to the well-behaved volume form of the inert background spacetime and by treating the particle as of arbitrarily small volume and of correspondingly high density. Moreover, as coordinate differences correspond to physical distances in this theory, we could always impose $m \to 0$ or $m \nrightarrow 0$ on the newtonian solutions in an equivalent manner.

But, this is not the situation with GTR. A particle is an *essential singularity* of the spacetime structure and no smooth spacetime can *replace* this essential singularity without changing the underlying physical basis of this theory.

Essentially, fluid approximation breaks down and there is no mathematical procedure to circumvent associated problems in the absence of "background" spacetime geometry to appeal to.

Furthermore, coordinate differences are not the geometrical or physical distances in this theory. If the density ρ is non-vanishing inside the source distribution then, for $m \to 0$, we have $r \to 0$ but, for $m \nrightarrow 0$, we have, $r \nrightarrow 0$. These two conditions are then the *coordinate freedom* in GTR.

But, it is customary to impose only the first of these two conditions while discarding the second as being unphysical. That is to say, it is customary to demand that the spacetime locations be *regular* in the sense that the orbits of the SO_3 group of rotations shrink to a zero radius ($r \to 0$) at every such location thereby discarding the situation of $r \nrightarrow 0$ as being unphysical. But, as the particle picture shows it, the condition that $r \nrightarrow 0$ is also *physically* motivated and allowed one.

Now, let us keep these difficulties aside and let us, to fix ideas, consider a spherical star using the fluid approximation. Then, we have some smooth spherical spacetime for this star.

Let us consider two copies of such a star separated by a very large distance today so that the evolution of one cannot essentially affect the evolution of the other. Evolution of the two copies may be expected, on the basis of general physical considerations, to be identical.

In the newtonian theory, this situation can be treated as a two-body problem with evolution of each star being independent of the other except for the, extremely weak, gravitational interaction of the two stars. The outcome of the collapse of each star is therefore identical here.

In General Relativity, however, the situation is more complicated than this. The main reason for the complication is that the spacetime of the two

stars taken together, no matter how far away the two stars are, is *not* globally spherically symmetric even though the spacetime of individual stars is globally spherically symmetric. Let us also ignore this problem, since our assumption, that the evolution of each star will be "weakly" affected by the other distant star, is still valid.

Let both these stars begin to collapse. Let us focus attention on one star. Then, for the newtonian and general relativistic situations, both, the presence of a very distant star should "weakly" affect the evolution of that star. So, if for our globally spherical spacetime, we had some definite outcome in the collapse of the star, there will be exactly the same outcome observable at locations of each star in the above situation.

We may add further copies of the same star, again separated by the *same* large distance from original stars and from each other. If every star collapsed to a definite outcome, there will be the same result observable at locations of each star in this, completely arbitrary, situation too.

Let us list our assumptions separately:

- the presence of a very distant star "weakly" affects the evolution of any particular star at a given epoch,
- the collapse of an individual star, for the spacetime of an individual star, results in some definite outcome, say, as a naked singularity or a black hole.

Then, if we have just one example of the gravitational collapse leading to a naked singularity for the spherical spacetime then, the Cosmic Censorship, that spacetime singularities are not visible to any observer, is certainly invalid!

Is this conclusion of the non-validity of the cosmic censorship really justified? That is to say, do we have faith in the above arguments in either the newtonian theory or in GTR?

In the newtonian theoretical framework, we have no reason to doubt this conclusion. Each star collapses to a point and we have the newtonian concept of a point-particle to back it up. We then know how to treat a point-particle in the newtonian theory. We then do not need any Cosmic Censor in Newton's theory.

Now, let us keep aside the question of the validity of the fluid approximation as we approach a spacetime singularity. Then, in GTR too, each spherical star collapses to a point-particle, that is, a spacetime singularity.

Now, we may find that in some situations a *null* trajectory emanates from this singularity, that is, the singularity is naked. Or, in some other situations, we may find that the singularity is covered by a *horizon*, that is, there is a black hole. That is, equivalently, a point-particle is naked or is covered by a horizon in GTR.

Then, since we have accepted a point-particle in GTR, we may renounce the Cosmic Censorship even in GTR and may accept the existence of naked singularities and black holes, both, as being physically or astrophysically relevant. Then, we choose to ignore all the serious problems of GTR as have been discussed earlier.

But, the situation with GTR is definitely not so simple and straightforward as it appears from the above discussion. The main reason is that a spacetime necessarily has *information* about the "past" and "future", both. That the two stars are separated by very large distance *today* does *not* mean that they were always so separated.

It could, for example, be that the two stars were "close" in the past but due to the directions of their respective velocities they moved away from each other to be separated by large distance at the present epoch.

What is *important* is that it is the *same spacetime* that is prescribed by GTR in either of these situations, of stars separated by some "small" distance or of stars separated by some very "large" distance between them. General Theory of Relativity, as a theory of gravitation, has no length-scale of any kind to distinguish, in any manner, between these two situations.

Now, let us return to the newtonian situation to consider an important related issue.

Notice then that the newtonian theory has *linear* laws. Such linear laws obey the superposition principle for their solutions. In any linear theory, particles must therefore be provided with *independent attributes* for any interactions and the corresponding laws for such interactions will also have to be mutually independent. Therefore, such linear laws cannot intrinsically contain any assertions about the interactions of elementary bodies, the particles. Hence, if we aim for the unification of physical laws, the theory cannot be linear nor can it be derived from such linear laws.

Furthermore, the linearity of the newtonian laws is also the primary reason as to why we can "superimpose" the individual newtonian solutions of each star to obtain the solution for the situation of combined stars. This is also the primary reason as to why we can, equally well, treat the "weak" gravitational effects of the presence of another very distant star as a perturbation of the newtonian solution of the individual star.

But, the spacetime of combined stars is not obtainable as a superposition of two or more number of spherical spacetimes. GTR is an intrinsically nonlinear theory of gravitation. The solutions of

the highly nonlinear field equations of GTR do not follow the superposition principle, in any conceivable manner whatsoever.

Then, the "perturbations of spherical spacetime" cannot, fundamentally, be faithful to the physical situation of combined stars. The spacetime of combined stars may be approximated by the perturbations of the spherical spacetime of a single star, but the corresponding results of physical significance cannot be faithfully obtained from them as was the case with a linear theory. Needless to say here then, the "perturbations of spacetime" obey linear laws.

This peculiar situation in GTR means that the outcome of the gravitational collapse of an individual star in a spacetime of the individual star is *not necessarily obtainable* in the spacetime of combined stars. Without adequate investigations, it, therefore, prevents us from accepting the earlier arguments leading to non-validity of the Cosmic Censorship in GTR.

Moreover, "some properties" of the spacetime of an individual star will not necessarily be those of the spacetime of stars taken together. That is to say, non-linearity of the field equations of GTR does not permit us to conclude, without adequate justification or investigation, that the spacetime of combined stars has some properties similar to the spacetime of an individual star.

Hence, without adequate justification or investigation, we cannot conclude that any specific property, in particular, the presence of an event horizon, of the spacetime of an individual star is obtainable quite generally for the spacetime of combined stars. That is to say, we cannot similarly conclude that even black holes will form in the spacetime of combined stars.

General Theory of Relativity is an intrinsically non-linear theory of gravitation.

III. FIELD-PARTICLE DUALISM IN GENERAL RELATIVITY

Let us now, in the light of all the earlier issues, reconsider the field-particle dualism in General Theory of Relativity.

Firstly, in these discussions, GTR is being considered as a theory of the "pure" gravitational field, other fields of nature not being "incorporated" in the (Schwarzschild) geometry of a point-mass, it has only the "gravitational mass" attributed to it. That is, we consider that this (Schwarzschild) geometry describes only the "pure" gravitational field of a mass-point.

Then, since a physical particle has other attributes, its spacetime geometry should the less be viewed as a pure gravitational field, as described by the Schwarzschild geometry, the closer one comes to the position of a particle - that is again a spacetime singularity.

Now, the concept of a "pure" gravitational field is a reminiscent of the Newtonian concept of the gravitational field of a point particle.

In Newton's theory, the "gravitational mass" of a particle is just one among the possible attributes of a particle. Each attribute of a particle produces the corresponding field. In this linear theory, any other field attribute of a particle cannot be related to or be determined by its gravitational mass attribute. It's laws are linear laws and, hence, we have to separately postulate the laws of interactions among the particles. Separate fields obey separate, meaning mutually independent, laws in this theoretical framework.

That is why we can speak of a "pure" gravitational field and "other" fields in a linear theory as is Newton's theory. Mutual independency of the laws for different fields makes this possible. Notice that it is mutual independency that is crucial in this theoretical framework.

But, in GTR, every attribute of a particle affects its spacetime geometry. That is, associated with every attribute of a particle corresponding to any field of Nature, there must be an appropriate corresponding "effect" on the spacetime geometry of the particle. This is inevitable in an intrinsically non-linear theory as is GTR. [The well-known examples of Reissner-Nordstrom and Kerr-Newman spacetimes clearly show this to us.]

Clearly, ascribing attributes, other than gravitational mass, to a particle in GTR is then equivalent to changing the spacetime geometry of a point mass that "continues" to be a non-removable, essential, spacetime singularity.

That is to say, a particle or a mass-point remains a spacetime singularity even after we account for, let us say, all the possible attributes of other fields of Nature in this way.

Then, many fundamental problems associated with the existence of a spacetime singularity, that have been discussed earlier, are unavoidable for this field-particle dualism.

Then, strictly speaking, we must formulate "external" laws for interactions of particles. Further, the attributes of a particle, such as its mass or charge, are *additive* and, hence, the required "external" laws of interactions of particles will, necessarily, be *linear or additive* in all these attributes of a point-particle.

But, all this is certainly contrary to the physical basis of GTR! In essence, a point-particle is inherently inconsistent with GTR.

Therefore, in view of the fundamental problems discussed earlier, it is then essential to renounce the field-particle dualism in GTR. That is to say, we renounce the picture of a point-particle, a spacetime singularity in GTR.

It may be noticed now that any open or concealed increasing of the number of dimensions from four does not improve the above problematic situation with the field-particle dualism.

Again, a point-particle will be a curvature singularity of the higher dimensional manifold. Then, if we continue with the field-particle dualism, all the fundamental problems associated with the singularity of the higher-dimensional spacetime manifold will continue to haunt us in the same manner as holds for the four-dimensional case of General Relativity. It then seems highly unlikely that any higher dimensional theory will, while continuing with the field-particle dualism, be able to provide any "basis" for the physical phenomena than that already provided by GTR.

Thus, we then consider here only singularity-free, smooth, four-dimensional spacetime geometries in General Relativity. Many such spacetime geometries are obtainable in GTR.

Then, an important question is: which of these smooth spacetime geometries are to be considered physically relevant?

Before we consider this issue, it is instructive to consider, on the basis of very general physical considerations, the physical properties that any such spacetime geometry should possess.

Firstly, we note that, in a spherically symmetric such spacetime, we will necessarily have $m = 4\pi \int \rho r^2 dr \nrightarrow 0$ at the center of the spacetime since ρ is a non-vanishing function of the radial coordinate and the well-behaved volume-form of any such smooth spacetime allows us to "define" an equivalent, non-vanishing, "gravitational" mass at its central spatial location.

That is to say, in such a smooth spacetime geometry, we could always "treat a gravitational mass" as of arbitrarily small volume and of correspondingly high density as a useful approximation. Then, for such a spherically symmetric spacetime, we will have $m \nrightarrow 0$ and, hence, $r \nrightarrow 0$. This will be the situation even in a general, non-symmetric, such spacetime.

Secondly, we note that any physical object is to be treated as a "spatially" concentrated form of *energy* in such a smooth spacetime geometry. As this object moves in space, the energy-distribution changes in the space. Then, the "motion" of any object is "equivalent" to change in the energy-distribution in such a spacetime. As we "move" any given object in the space, the energy-distribution changes accordingly and, hence, energy-distribution should be arbitrary if a physical body moves so in such a spacetime.

Remarkably, this picture is also consistent with the following observation that can be said to form the basis for cosmology.

Gravitational and other fields have been rearranged on the Earth on many instances. That is to say, the local distributions of different fields of Nature have been (and can be) altered on the Earth in an *arbitrary* manner at any time.

Provided that the Earth is not any special location in the universe, the *Copernican principle*, the local source distribution of different fields of Nature can be altered in an *arbitrary* manner *anywhere and at any time* in the universe. This would be possible if the distribution of fields were arbitrary in a spacetime continuum.

Clearly, the above two entirely different approaches reach the same conclusion in the form of the *required spacetime*. This cannot be just a mere simple coincidence.

As a matter of principles, we should then not consider an *isolated object, or its spacetime, or the outcome of its collapse*, as being physically relevant since no object is genuinely isolated in the universe. The non-linearity of the field equations of GTR then clearly indicates that such spacetime geometries will get *modified* when we embed the corresponding objects in the universe. Properties of the modified spacetime geometry will not necessarily be those of the spacetime of an isolated object. Thus, the spacetime of an isolated physical object cannot be a physically relevant or meaningful spacetime geometry.

Therefore, the issue of cosmic censorship for such spacetimes of isolated or individual physical objects like an isolated star is not any relevant issue for the physical theories.

Furthermore, as a matter of principles again, we should similarly not consider any smooth, singularity-free spacetime geometry that does not, without changing its global mathematical properties, let us "shift physical objects" arbitrarily in it to be physically relevant or meaningful. We do shift ourselves arbitrarily in the "spacetime geometry of the universe" we live in.

The non-linearity of the field equations of GTR then clearly indicates in this case that such spacetimes will get *modified globally* when we "shift" physical objects in them. Again, the nonlinearity of the field equations of General Relativity then indicates that the global properties of the modified spacetime will not necessarily be those of the original spacetime.

Therefore, if in a chosen spacetime of the universe, we cannot "shift" physical objects *arbitrarily* in a local spatial region without changing global

spacetime properties, it is difficult to imagine as to how that spacetime can faithfully represent the *observable universe*.

One now sees clearly the emerging confluence of the above two approaches. The two approaches demand the existence of the *same*, mathematically smooth, spacetime geometry.

Therefore, the geometric properties of an appropriate smooth, singularity-free spacetime, one with arbitrary spatial properties, are the only that are genuinely physically relevant ones. This is, of course, subject only to the caveat that the theoretical description of physical systems based on any geometric continuum is permissible.

As this spacetime is *singularity-free* everywhere, its initial data is singularity-free or *regular* everywhere on its spacelike hyper-surfaces. The question of cosmic censorship then reduces to whether any observable spacetime singularity results in temporal evolution of this initial data. The issue of cosmic censorship, if any, can then be settled only when we have, explicitly, such a spacetime geometry to study.

Hence, only if the naked singularities or event horizons arise for such a smooth spacetime, can these be physically relevant concepts.

IV. BEYOND STANDARD FORMALISM OF GENERAL RELATIVITY

We thus consider here only smooth spacetime geometries in GTR. Moreover, any geometric theory of gravitation, as is GTR, must then incorporate all the permissible field-attributes of various possible fields of Nature in some mathematically smooth spacetime geometry. Thus, there must then exist a spacetime incorporating all the possible fields of Nature in it. After all, the fields are continuous, live in space and evolve in time.

There must therefore exist a spacetime in such a theory that has all the "fields" of Nature incorporated in its geometry. Such a spacetime can be considered to provide the geometry of the *total field*, in Einstein's sense [1].

Consequently, some general relativistic spacetime must then be a *complete spacetime geometry* in which "particles" themselves, as concentrated form of energy but not as spacetime singularities, would *everywhere in space* be describable as singularity-free.

Provided, of course, that the theoretical description of "physical systems" is permissible on the basis of a geometric spacetime continuum, as is the underlying basis of the General Theory of Relativity, this is then the logically inevitable conclusion. Such a spacetime then admits "arbitrary" distribution of fields in it.

The required spacetime that is everywhere regular and singularity-free has a (pseudo-)metric of the form:

$$\begin{aligned} ds^2 \;=\; & -P^2Q^2R^2dt^2 + \gamma^2 P'^2 Q^2R^2B^2dx^2 \\ & + \gamma^2P^2\bar{Q}^2R^2C^2dy^2 \\ & + \gamma^2P^2Q^2\tilde{R}^2D^2\,dz^2 \end{aligned} \tag{1}$$

where $P \equiv P(x)$, $Q \equiv Q(y)$, $R \equiv R(z)$, $B \equiv B(t)$, $C \equiv C(t)$, $D \equiv D(t)$ and γ is a constant. Also, $P' = dP/dx$, $\bar{Q} = dQ/dy$ and $\tilde{R} = dR/dz$.

Now, as can also be checked easily, the metric (1) admits, precisely, three spacelike homothetic Killing vectors. From Lie's theory of differential equations, it then follows that there will be three *arbitrary* functions of spatial coordinates corresponding to these three spacelike homothetic Killing vectors. These are the spatial functions P, Q, R. We also note here that the spacetime of (1) is also a machian spacetime [2, 3, 4].

In general, there are two types of curvature singularities of the metric (1). Singularities of the first type, vanishing of any temporal functions B, C, D, are singular hyper-surfaces while singularities of the second type, when any one of the spatial functions P, Q, R is vanishing, are singular spatial data for (1).

The locations for which the spatial derivatives vanish are, however, coordinate singularities. The curvature invariants of (1) do not blow up at such locations. There are also obvious degenerate metric situations when any of the spatial functions is infinite for some range of the coordinates.

We may now consider an appropriate energy-momentum tensor for the fluid in the spacetime and write down the field equations of GTR. But, as can be verified, the field equations do not determine the spatial functions P, Q, R.

The energy-density in the spacetime of (1) varies as $\rho \propto 1/P^2Q^2R^2$ and is *arbitrary* since the field equations do not determine the spatial functions P, Q, R.

But, it is known [5] that the energy-momentum tensor is not a satisfactory concept in General Relativity [27]. Reason for this appears to be its dependence on the concept of a point-particle.

Recall that to obtain the energy-momentum tensor we consider a collection of particles and try to obtain a suitable *continuum approximation* of different physical quantities such as energy density, momentum flux etc.

In the newtonian situation this procedure works well because the background spacetime geometry is non-singular.

But, in the standard approach to GTR, wherein we consider each spacetime geometry as a solution of the Einstein field equations to be of relevance to some, hypothetical or not, physical situation, the spacetime singularities do not permit us any meaningful averaging procedure, as was seen earlier. Consequently, various difficulties in defining the energy-momentum tensor arise in this situation and doom the use of the energy-momentum tensor to be a failure.

Thus, the problem, now, is that the energy-momentum tensor is, in general, not a well-defined concept. Therefore, we are not justified in using it to obtain the Einstein field equations. Then, we do not "know" for sure as to how to determine the three temporal functions in the metric (1) completely. We therefore have no tools at hands to construct the four-dimensional spacetime geometry of the metric (1).

That is why, we essentially abandon the four-dimensional picture and consider only a smooth, three-dimensional manifold admitting a positive-definite metric of the form (2). (See below.)

What is now so crucial to realize is that there are three temporal functions B, C, D in the metric corresponding to "motion" along three spatial directions. Consequently, we may then consider a physical object (on any $t =$ constant hyper-surface of this spacetime) and consider that it "moves" with velocity having appropriate components along the three spatial directions.

Thus emerges the picture of "motion" in this spacetime that motion of a physical object corresponds only to a change in the energy density in this spacetime.

Therefore, we essentially abandon the four-dimensional spacetime as a basic picture and consider only a smooth, three-dimensional manifold admitting a positive-definite metric of the form

$$\begin{aligned} d\ell^2 \;=\; & P'^2 Q^2 R^2\, dx^2 \\ & + P^2 \bar{Q}^2 R^2\, dy^2 \\ & + P^2 Q^2 \tilde{R}^2\, dz^2 \end{aligned} \qquad (2)$$

where $P(x)$, $Q(y)$, $R(z)$ are arbitrary functions of their respective arguments. We shall call the space of the metric (2) as the *base space* or *Einstein space*. It will be denoted by the symbol $\mathbb{B}$.

Any specific choice of functions, say, P_o, Q_o, R_o gives us a specific spatial distribution of energy ρ in the space of (2). As "concentrated" energy "moves" in the space, we have the original set of functions changing to a "new" set of corresponding functions, say, P_1, Q_1, R_1.

Spatial functions P, Q, R constitute *initial data* for the base space $\mathbb{B}$. Then, "motion" as described above is, basically, a *change of one set of initial data* to *another set of initial data* with "time". Therefore, we will be considering "motion" as a *mapping from the initial data set to the initial data set* of the base space $\mathbb{B}$ [28].

Clearly, we are considering isometries of the metric (2) while considering "motion" of this kind. That we will remain within the group of the isometries of (2) is always "guaranteed" as long as we are restricting to the triplet of functions P, Q, R subject, of course, to some other further conditions on them obtainable as follows.

Since the vanishing of any of the spatial functions P, Q, R is a *curvature singularity* of the metric (2), we will have to restrict ourselves to non-vanishing, strictly positive (or strictly negative), real-valued such functions of their respective arguments. This type of a restriction is necessary and sufficient, both, to remain within the group of isometries of (2).

Let us then denote by $\mathcal{F}$ the set of all strictly positive, nowhere-vanishing, real-valued functions from $\mathbb{R} \to \mathbb{R}^+$. Now, consider a direct-product $\mathcal{F}\times\mathcal{F}\times\mathcal{F}$ defined by an ordered triplet $(P, Q, R) \in \mathcal{F} \times \mathcal{F} \times \mathcal{F}$. As can be easily verified, this set, $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$, is a *convex* set.

Now, consider the set of all the possible mappings from $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathbb{R}$. Every such map defines a family of "curves" connecting points of $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$ and, hence, it defines "motion" in our picture here. Every such map is then a *suitably defined dynamical system* describing motion in the base space $\mathbb{B}$. (This notion of a dynamical system will be made mathematically precise later at a suitable stage.)

Then, different "motions" of any "specific region of concentrated energy" are "classifiable" in terms of the corresponding classification of suitably defined dynamical systems.

Therefore, what we are looking for is a *complete classification of such suitably defined dynamical systems* definable on the basic product set $\mathcal{F} \times \mathcal{F} \times \mathcal{F}$ or the base space $\mathbb{B}$.

V. MATHEMATICAL REQUIREMENTS

A. Mathematical short-forms

In what follows, we shall adopt the following short-forms. We shall write for "with respect to" as "wrt". We also write "s.t." for "such that" and "a.e." for "almost everywhere" which means that the property holds except for a set of measure zero. We will also write "SBS" for "Standard Borel Space". Different such short-form notations have been listed in the Appendix.

B. Formalism for dynamics

In this connection, we firstly note that the differential of the volume-measure on the base space $\mathbb{B}$ defined by metric (2) is

$$d\mu \;=\; P^2Q^2R^2\left(\frac{dP}{dx}\frac{dQ}{dy}\frac{dR}{dz}\right)\,dx\,dy\,dz \qquad (3)$$

This differential of the volume-measure vanishes when any of the derivatives, of P, Q, R with respect to their arguments, vanishes. (Recall that P, Q, R are non-vanishing over $\mathbb{B}$.)

We, therefore, define a set, to be called a **P-set**, denoted as P, by the following definition:

> A **P-set** of the base space $\mathbb{B}$ is the interior of a region of $\mathbb{B}$ for which the differential of the volume-measure, (3), is vanishing on its boundary while it being non-vanishing at any of its interior points.

Furthermore, each P-set is a metric space with a metric

$$d\ell^2 \;=\; Q^2R^2dP^2 + P^2R^2dQ^2 + P^2Q^2dR^2$$

Note that all the spatial functions P, Q, R are *monotonic* within a P-set.

Now, any two given P-sets, P_i and P_j, $i, j \in \mathbb{N}$, $i \neq j$, are, consequently, *disjoint sets.* For, if the intersection of P_i and P_j were non-empty, there would be *interior* points of each of the two P-sets at which the differential of volume-measure, (3), would be vanishing, contrary to our definition of a P-set as given above.

Now, for a given, specific choice of, spatial functions P, Q, R, *ie*, for a point of $\mathcal{F}\times\mathcal{F}\times\mathcal{F}$, the base space $\mathbb{B}$ is, uniquely, expressible as a countably infinite union of the *closures* of P-sets, *ie*, $\mathbb{B} = \bigcup_{i=1}^{\infty}(P_i)^c$.

Furthermore, there are uncountably many such families of P-sets with the base space $\mathbb{B}$ being equal to the countably infinite union of the closures of the members of each such family of P-sets. Each such family of P-sets is characterized by an ordered triplet $(P, Q, R) \in \mathcal{F}\times\mathcal{F}\times\mathcal{F}$.

Hence, "motion" in the present picture can also be looked upon as a "suitably defined map" from one such family of P-sets to another such family of P-sets.

A concentrated form of energy is a P-Set within this framework. The reason for naming the set under consideration as a P-set is then clear. A **P-set** is a *physical particle* in the present scenario. That is to say, we can associate physical attributes of a physical particle with a P-set. (Then, *attributes* of a physical particle will be *measures* definable on a P-set. We will be dealing with measures is then clear.)

Interestingly, and crucially, any P-set is *not* contractible to any of its points as a P-set. That is to say, a singleton subset $\{\{x\} : x \in \mathbb{B}\}$ of $\mathbb{B}$ is a *not* a P-set.

Each P-set is then imagined to be a *physical particle.* That is to say, each P-set can be ascribed different properties of a physical particle. Then, since a P-set cannot be shrunk to a point as a P-set of the base space, each P-set is a physical particle which is, fundamentally, an extended basic physical body in this picture.

The question then arises about an appropriate mathematical description of the dynamics of such particle systems.

As a concrete example of the current situation, let us imagine that the 3-dimensional space (of finite or infinite total volume, depending on the functions P, Q, R) is uniformly filled with *solid balls* touching each other. There will, of course, be *gap-filling spaces* in such a distribution which will be having the same shape at all the corresponding locations. Then, let us mark one solid ball as *Red* and another "distant" (meaning not touching the first one) as *Green.*

We now want to mathematically describe the motion *of the* Red ball *so that it* moves towards *the* Green ball*, touches that ball and* returns *to its original position.* Clearly, the gap-filling spaces must change their shapes so that the Red ball performs this motion. Moreover, even the balls may change their shapes to move.

In the above situation, Red/Green ball is one type of a P-set and Gap-filler space is the other type of a P-set. Any "motion" is then possible iff the P-sets change their shapes and adjust themselves as they perform the "motion".

Going further with this example, we may also consider that two or more balls (some mutually touching each other) "stick to each other" as they "move collectively". Again, some of the P-sets must change their shapes suitably for such a motion to be possible at all.

Therefore, a P-set may also change its *shape* during its dynamical evolution. In fact, some P-set(s) must change their *shapes* if at all there is to be "motion" in the present picture. Then, this must also reflect in the mathematical description being used in the present picture.

We, once again, stress that any point of the base space $\mathbb{B}$, as a singleton set, is *not* a P-set. Hence, a particle cannot be a point within this picture and is always an extended body. Furthermore, a physical body is, in general, a collection of such basic extended particles.

Hence, any such mathematical description must also be applicable to a collection of P-sets since a physical body is imagined to be a collection of P-sets and such a physical body "may move" while maintaining its "shape".

In what follows, we review [6, 7, 8, 9] some mathematical aspects of relevance to the present physical framework. Much of it is well-known (in particular, from the ergodic theory). Nonetheless, for completeness, for fixing of notations and for the benefit of those not familiar with the involved terminology, we include it here.

Now, we recall that a *metric space* is a pair (X, d) where X is a set, $d : X \times X \to \mathbb{R}$ is a real-valued function, called the *distance function*, that satisfies, $\forall\ x, y, z \in X$, properties

(a) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$

(b) $d(x, y) = d(y, x)$ (Symmetry property)

(c) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

Moreover, a *pseudo-metric space* is a pair (X, ℓ) where X is a set, $\ell : X \times X \to \mathbb{R}$ is a function, called the *pseudo-distance function*, that satisfies, $\forall\ x, y, z \in X$ (1') $\ell(x, y) \geq 0$ and $\ell(x, x) = 0$ and properties (2) and (3) of the distance function mentioned above.

Furthermore, there exists a canonical way of obtaining a metric space from a pseudo-metric space. Consider a pseudo-metric space (X, ℓ). Define on X an equivalence relation, denoted by $\sim$ here, s.t. $x \sim y$ iff $\ell(x, y) = 0$. Now, let us denote by Y the set of all equivalence classes of X under the equivalence relation $\sim$. Further, define for $A, B \in Y$ a function $e(A, B) = \ell(x, y)$ where $x \in A$ and $y \in B$. Then, the function $e : Y \times Y \to \mathbb{R}$ is a distance function on Y. Further, let $\Pi : X \to Y$ be the natural projection, *ie*, for $x \in X$, $\Pi(x) = \{y \in X | x \sim y\}$ an equivalence class containing x. This projection function Π is an isometry, that is to say, it preserves the distance function e.

Then, it is now important to note that the *base space* $\mathbb{B}$ *is a pseudo-metric space (and not a metric space)* admitting a pseudo-metric (2), to be called as the *Einstein pseudo-metric*, with the functions P, Q, R being nowhere vanishing functions of their respective arguments, and x, y, z providing the "coordinatization" of $\mathbb{B}$ s.t. $\mathbb{B}$ is a 3-dimensional pseudo-Riemannian manifold.

Moreover, the restriction of the Einstein pseudo-metric (2) on a P-set is a distance function. *Therefore, we may, equivalently, define a P-set as that region of the base space* $\mathbb{B}$ *over which the restriction of the Einstein pseudo-metric (2) is a metric function.* It is now obvious as to why a singleton set of $\mathbb{B}$ cannot be a P-set.

As an aside, we note also that any averaging procedure is now well-defined over a collection of P-sets. Therefore, we may, in a mathematically meaningful way, talk about the "concept of energy-momentum tensor" and some relation between the averaged quantities, an "equation of state" defining appropriately the "state of a physical system" under consideration. Clearly, different such physical concepts remain "mathematically meaningful" in the present situation [29]. But, we shall not consider this issue in the present paper.

C. Physical space is Standard Borel

Let us denote by d the distance function obtainable in a canonical manner from the Einstein pseudo-metric (2).

Let us now consider the Einstein metric space $(\mathbb{B}, d)$. A set $B_d(x_o, r) = \{x \in \mathbb{B} \,|\, d(x, x_o) < r\}$ is called as an *open* r*-ball around fixed* $x_o \in \mathbb{B}$, where r is a positive real number.

We call a subset $A \subset B$ as *bounded* if $d/A \times A$ is a bounded function. We call the real number $\sup\{d(x, y) \,|\, x, y \in A\} = \delta(A)$ as a *diameter of a non-empty bounded set* A. A subset $A \subset \mathbb{B}$ is called an *open subset* of $\mathbb{B}$ if $\forall\ x_o \in A$, there exists a positive real number r s.t. $B_d(x_o, r) \subset A$. The collection Γ of all open subsets of $\mathbb{B}$ then provides the *metric topology* on $\mathbb{B}$.

It is now easy to see that every P-set is a member of the metric topology Γ since a suitable open r-ball around each of its points is contained in it and, hence, every P-set is open in $(\mathbb{B}, \Gamma)$. However, note that every open set of $(\mathbb{B}, \Gamma)$ is *not* a P-set of $(\mathbb{B}, d)$ and that "being a P-set of the Einstein space $\mathbb{B}$" is *not* a *topological property.*

Now, it is also easy to see that the space $\mathbb{B}$ is a separable and compact metric space. But, every compact metric space is complete [6]. Hence, the base space $\mathbb{B}$ is a complete separable metric space. Note further that $\mathbb{B}$ is uncountable.

Recall that a Topological Space (X, Υ) is called *Polish* if there exists a metric d on a non-empty set X s.t. the metric space (X, d) is complete and separable, and the topology Υ, called a *Polish Topology*, is induced by the metric d.

Now, by a σ*-algebra* $\mathcal{B}$ we mean a non-empty collection $\mathcal{B}$ of subsets of X which is closed under countable unions and complements.

The intersection of any family of σ-algebras on X is again a σ-algebra. If $\mathcal{E}$ is any collection of the subsets of X, then the intersection of all the σ-algebras containing $\mathcal{E}$ is the smallest σ-algebra containing $\mathcal{E}$ and is called the σ*-algebra generated by the collection* $\mathcal{E}$.

A *Borel Structure (Borel space)* is a pair $(X, \mathcal{B})$ where X is a non-empty set and $\mathcal{B}$ is a σ-algebra of the subsets of X. If A is a non-empty subset of X, then the collection of all subsets of X of the form $A \bigcap B,\ B \in \mathcal{B}$ is a σ-algebra on A, called as the *induced σ-algebra on* A, which we denote by $A \bigcap \mathcal{B}$ or by $\mathcal{B}|_A$ or by $\mathcal{B}/A$.

Two Borel spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are (Borel) isomorphic to each other if $\exists$ a one-one map ϕ of X_1 onto X_2, a *Borel isomorphism* of two Borel spaces, s.t. $\phi(\mathcal{B}_1) = \mathcal{B}_2$.

A Borel space Borel-isomorphic to the Borel space of an uncountable complete separable metric space is called a *Standard Borel Space* (SBS). A SBS is Borel-isomorphic to the Borel space of the unit interval, $[0, 1]$, equipped with the σ-algebra generated by its usual topology.

A SBS, $(X, \mathcal{B})$, is therefore a Polish Topological Space with the σ-algebra generated by open sets in X. We call this σ-algebra the Borel σ-algebra of X and we often denote it as $\mathcal{B}_X$. Members of $\mathcal{B}_X$ are the *Borel sets* of X.

Now, as noted before, the Einstein space $(\mathbb{B}, d)$ is an uncountable, complete, separable metric space. Therefore, we are dealing here with a *Polish Topological Space* [7, 8, 10] $(\mathbb{B}, \Gamma)$ where Γ is the topology defined earlier.

Then, if we construct the smallest σ-algebra of the subsets of $\mathbb{B}$ that contains every open subset of $\mathbb{B}$, that is to say, the Borel Set Structure on $\mathbb{B}$, then we will be dealing with a SBS - the *Standard Borel Space* - $(\mathbb{B}, \mathcal{B})$. The Einstein-space or the physical space $\mathbb{B}$ is therefore a SBS.

VI. SOME RELEVANT KNOWN RESULTS

Set theoretic considerations

In this connection, we firstly note the following results from the descriptive set theory [8]:

- Any set in $\mathcal{B}_{\mathbb{B}}$ is either countable or has the cardinality $\mathbf{c}$ of the continuum.
- If $A, B \in \mathcal{B}_{\mathbb{B}}$ are of the *same cardinality*, then A and B are Borel isomorphic.
- If Y is another complete separable metric space of the *same cardinality* as $\mathbb{B}$ and $\mathcal{B}_Y$ is its Borel σ-algebra, then $(\mathbb{B}, \mathcal{B}_{\mathbb{B}})$ and $(Y, \mathcal{B}_Y)$ are Borel isomorphic.
- From the above, it follows that if $A \in \mathcal{B}_{\mathbb{B}}$ and $C \in \mathcal{B}_Y$ have the *same cardinality*, then the Borel spaces $(A, A \bigcap \mathcal{B}_{\mathbb{B}})$ and $(C, C \bigcap \mathcal{B}_Y)$ are Borel isomorphic.

Note also that the restriction of $\mathcal{B}$ in $(\mathbb{B}, \mathcal{B})$ to any of the Borel sets $A \in \mathcal{B}$ yields again a new Borel space $(A, \mathcal{B}/A)$.

Then, two P-sets of the *same cardinality*, belonging either to the same Einstein-space $(\mathbb{B}, d)$ or to two different Einstein-spaces $(\mathbb{B}, d_1)$ and $(\mathbb{B}, d_2)$, are Borel-isomorphic. Further, the restriction of $\mathcal{B}_{\mathbb{B}}$ to a P-set yields a Borel space.

Note that a P-set is to be a physical particle and the physical attributes of a particle are to be measures on a P-set. Surely, any copies of a physical particle must be indistinguishable from each other except for their spatial locations.

Then, the Borel-isomorphic equivalence of P-sets of the *same cardinality* is consistent with relevant such physical conceptions since the same values of measures exist on equivalent such P-sets. This is then an indication of the internal consistency of the present physical framework.

Now, let $\mathcal{N}$ be a collection of subsets of space $\mathbb{B}$, $\mathcal{N} \subseteq \mathcal{B}$, s.t.

1. $\mathcal{N}$ is closed under countable union
2. $A \in \mathcal{B}$ and $N \in \mathcal{N}$ implies that $A \bigcap N \in \mathcal{N}$,
3. for $N \in \mathcal{N},\ N^c = \mathbb{B} - N \notin \mathcal{N}$.

We call $\mathcal{N}$ the *σ-ideal*. Note that $\mathcal{N}$ may contain either $\emptyset$ or $\mathbb{B}$, but not the both.

If $\mathcal{E}$ is any collection of subsets of $\mathbb{B}$, then there exists a smallest σ-ideal containing $\mathcal{E}$, the intersection of all σ-ideals containing $\mathcal{E}$. It is called the *σ-ideal generated by* $\mathcal{E}$ and is obtained by taking all sets of the form $B \bigcap E$ with $B \in \mathcal{B}$, $E \in \mathcal{E}$ and taking countable unions of such sets.

A subset E of $\mathbb{B}$ is said to have the *property of Baire* if E can be expressed as a symmetric union of an open set G and a set M of the first Baire category, *ie*, expressible as the union $E \equiv G \triangle M = (G - M) \bigcup (M - G)$. If E has the property of Baire, then so does its complement in $\mathbb{B}$. A set of the first Baire category is a countable union of nowhere dense sets.

Since $\mathbb{B}$ is a SBS, every Borel subset of $\mathbb{B}$ has the property of Baire since the σ-algebra of sets with the property of Baire includes the Borel σ-algebra of $\mathbb{B}$. The collection of subsets of $\mathbb{B}$ with the property of Baire is a σ-algebra generated by open subsets together with the subsets of the first Baire category. Subsets of first Baire category in $\mathbb{B}$ form a σ-ideal in the σ-algebra of sets with the property of Baire.

In particular, we note that every P-set is a Borel subset of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ and, hence, has the property of Baire. (When we want to emphasize that the base space has metric d, we will use the notation $(\mathbb{B}, \mathcal{B}, d)$.)

For $A, B \in \mathcal{B}$, we write $A = B \ (\mathrm{mod}\,\mathcal{N})$ if $A - B$ and $B - A$, both, belong to $\mathcal{N}$.

A subset $B \subset \mathbb{B}$, $B \in \mathcal{B}$, is said to be *decomposable* if it is expressible as a union of two disjoint sets from $\mathcal{B} - \mathcal{N}$. Clearly, every such decomposable set belongs to $\mathcal{B} - \mathcal{N}$.

We say that the Borel σ-algebra $\mathcal{B}$ of subsets of $\mathbb{B}$ satisfies the *countability condition* if every collection of pairwise disjoint sets from $\mathcal{B} - \mathcal{N}$ is either finite or countably infinite.

Now, let X be a set, $\mathfrak{P}(X)$ be its *power set* (the set of all subsets of X) and $\mathcal{A}$ an algebra of subsets of X. A *set function* $m : \mathfrak{P}(X) \to \mathbb{R}$, defined on $\mathcal{A}$, is a *finitely additive measure* if

(1) $0 \le m(A) \le \infty \ \forall \ A \in \mathcal{A}$

(2) $m(\emptyset) = 0$

(3) $m(A \bigcup B) = m(A) + m(B)$ if $A, B \in \mathcal{A}$ and $A \bigcap B = \emptyset$.

A finitely additive measure m with an additional property that

(4) $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m\left(A_i\right)$ for every pairwise disjoint sequence $\{A_i : i \in \mathbb{N}\}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

is called a *Countably Additive Measure* or simply *Measure*. Further, if all subsets of sets of measure zero are measurable, a measure is said to be a *complete measure*.

If m is a measure on $(X, \mathcal{A})$, then a set $E \in \mathcal{A}$ is of *finite m-measure* if $m(E) < \infty$; is of *σ-finite m-measure* if $\exists \ \{E_i\}, \ i \in \mathbb{N}, \ E_i \in \mathcal{A}$ s.t. $E \subseteq \bigcup_{i=1}^{\infty} E_i$ and $m(E_i) < \infty, \ \forall \ i \in \mathbb{N}$. If $m(A), \ A \in \mathcal{A}$ is finite (σ-finite) then the measure m is *finite (σ-finite) measure on $\mathcal{A}$*. A measure is *totally finite or totally σ-finite* if $m(X)$ is finite or σ-finite.

If $\mathcal{A}$ is a σ-algebra of subsets of X and m is a measure on it, then $(X, \mathcal{A})$ is a *measurable space* and the members of $\mathcal{A}$ are *$\mathcal{A}$-measurable sets*. We call $(X, \mathcal{A}, m)$ as a *Measure Space*.

We also define a *signed measure* as an extended, real-valued, countably additive set function μ on the class, $\mathcal{A}$, of all measurable sets of a measurable space $(X, \mathcal{A})$ s.t. $\mu(\emptyset) = 0$, and s.t. μ assumes at most one of the values $+\infty$ and $-\infty$.

If μ is a signed measure on a measurable space $(X, \mathcal{A})$, we call a set E *positive (negative) wrt* μ if, $\forall \ F \in \mathcal{A}$, $E \bigcap F$ is measurable and $\mu(E \bigcap F) \ge 0$ ($\mu(E \bigcap F) \le 0$). The empty set is both μ-positive and μ-negative in this sense.

If μ is a signed measure on $(X, \mathcal{A})$, then there exist two disjoint sets $A, B \in \mathcal{A}$ s.t. $A \bigcup B = X$ and A is μ-positive while B is μ-negative. The sets A and B are said to form the *Hahn Decomposition of X wrt μ*. It is not unique.

For every $E \in \mathcal{A}$, we define $\mu^+(E) = \mu(E \bigcap A)$, the *upper variation of* μ, and $\mu^-(E) = \mu(E \bigcap B)$, the *lower variation of* μ, and $|\mu|(E) = \mu^+(E) + \mu^-(E)$, the *total variation of* μ, where A, B are as in the Hahn decomposition. [Note that $|\mu(E)|$ and $|\mu|(E)$ are not the same.]

The upper, the lower and the total variations (of μ) are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$ $\forall \ E \in \mathcal{A}$, the *Jordon decomposition*. If μ is finite or σ-finite, then so are μ^+ and μ^-; at least one of μ^+ and μ^- is always finite.

A *simple function on* $(X, \mathcal{A})$ is $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ where $E_i \in \mathcal{A}$, χ_{E_i} is the *characteristic function of the set* E_i and $\alpha_i \in \mathbb{R}$. This simple function f is *μ-integrable* if $\mu(E_i) < \infty \ \forall \ i$ for which $\alpha_i \neq 0$. The *μ-integral of* f is $\int f(x) d\mu(x)$ or $\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)$.

If, $\forall \ \epsilon > 0$, $\lim_{n \to \infty} m\left(\{x \in X : |f_n(x) - f(x)| \ge \epsilon\}\right) = 0$, a sequence $\{f_n\}$ of a.e. finite-valued measurable functions is said to *converge in measure* to a measurable function f.

Given two integrable simple functions f and g on a measure space $(X, \mathcal{A})$, define now a pseudo-metric $\rho(f, g) = \int |f - g| \, d\mu$. A sequence $\{f_n\}$ of integrable simple functions is *mean fundamental* if $\rho(f_n, g_m) \to 0$ if $n, m \to \infty$.

An a.e. finite-valued, measurable function f on $(X, \mathcal{A})$ is *μ-integrable* if there is a mean fundamental sequence $\{f_n\}$ of integrable simple functions which converges in measure to f.

Lebesgue-Radon-Nikodym (LRN) theorem [7, 9] states that: If $(X, \mathcal{A}, m)$ is a totally σ-finite measure space and if a σ-finite measure ν on $\mathcal{A}$ is absolutely continuous wrt m, then $\exists$ a finite valued measurable function f on X s.t. $\nu(E) = \int_E f \, d\mu$ for every measurable set $E \in \mathcal{A}$. The function f is unique in the sense that if also $\nu(E) = \int_E g \, d\mu$, then $f = g \ (\mathrm{mod}\ \mu)$, *ie*, equality holding modulo a set of μ-measure zero or μ-a.e.

If μ is a totally σ-finite measure and if $\nu(E) = \int_E f \, d\mu \ \forall \ E \in \mathcal{A}$, we write $f = \frac{d\nu}{d\mu}$ or $d\nu = f \, d\mu$. We call $\frac{d\nu}{d\mu}$ the LRN-derivative and all the properties of the differential formalism hold for it, importantly, μ-a.e.

A set $E \subseteq X$ is G_δ if $E = \bigcap_{i=1}^{\infty} U_i, \ U_i$ open in X; it is F_σ if $E = \bigcap_{i=1}^{\infty} C_i, \ C_i$ closed in X. Class of all G_δ (F_σ) sets is closed under finite unions and countable intersections.

For a locally compact Hausdorff space X, let $\mathcal{C}$ be the class of all compact subsets of X, $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$, and $\mathcal{U}$ be the class of all open sets belonging to $\mathcal{A}$. Also, let $\mathcal{C}_o$ denote the class of all compact subsets of X which are G_δ, $\mathcal{A}_o$ be the σ-algebra generated by $\mathcal{C}_o$, and $\mathcal{U}_o$ be the class of all open sets belonging to $\mathcal{A}_o$.

Members of $\mathcal{A}_o$ are the *Baire sets of* X and $\mathcal{U}_o$ is the class of all open Baire sets. A real-valued function on X is *Baire measurable* or simply a *Baire function* if it is $\mathcal{A}_o$-measurable. A measure on $\mathcal{A}_o$ is called a *Baire measure*. If $\mathcal{A} = \mathcal{B}_X$, then the members of $\mathcal{A}$ are *Borel measurable* and measure on $\mathcal{A}$ is a *Borel measure*. A real-valued $\mathcal{B}_X$-measurable function is called a Borel-measurable function or simply a *Borel function*.

A *Haar measure* is a Borel measure μ in a locally compact topological group X s.t. $\mu(U) > 0$ for every non-empty Borel open subset U of X, and $\mu(x\,E) = \mu(E)$ for every Borel set E of X. A Haar measure is a translation-invariant Borel measure which is not identically zero.

An *atom of a measure* m is an element $E \in \mathcal{A}$, $E \neq \emptyset$ s.t. if $F \subset E$, then either $m(F) = 0$ or $m(F) = m(E)$. A measure with no atoms is *non-atomic*. A measure space with non-atomic measure on it is non-atomic. We also call a finite measure ν on $\mathcal{A}$ *purely atomic* if there exists a countable set $C \in \mathcal{A}$ s.t. $\nu(X - C) = 0$.

A measure m is called *regular* if given $A \in \mathcal{A}$ and $\epsilon > 0$, there exist $G, F \in \mathcal{A}$ s.t. $m(G) - \epsilon \leq m(A) \leq m(F) + \epsilon$.

For $(X, \mathcal{A}, \mu)$ a measure space, let $\mathfrak{S}(\mu)$ be the set of all measurable sets with finite μ-measure. For any $E, F \in \mathfrak{S}(\mu)$, let $\rho(E, F) = \mu(E \triangle F)$. The function ρ so defined is a metric on $\mathfrak{S}(\mu)$ and the metric space $(\mathfrak{S}(\mu), \rho)$ is called the *metric space of or associated to* $(X, \mathcal{A}, m)$.

A SBS equipped with a finite or σ-finite measure m is called as the *Standard Measure Space* (SMS). If, in particular, $m(X) = 1$, a SMS is called as the *Standard Probability Space* (SPS) and m is called as the *probability measure on* $\mathcal{B}$.

Now, we note that a P-set is an uncountable separable complete metric space with the restriction of the Einstein pseudo-metric (2) to it as a metric. Therefore, Borel-isomorphic P-sets of the *same cardinality* are Borel-isomorphic SMSs/SPSs. Such P-sets, as particles of the *same species*, then possess measures as indistinguishable physical attributes.

Indeed, it can be called a deep mystery of the micro-cosmos as to how all the elementary particles of the same type possess exactly identical physical attributes such as mass, charge etc. Perhaps, the above characterization of physical particles as P-sets of the Einstein space $(\mathbb{B}, d)$ is a solution to this ill-understood deep mystery.

In the above, we dealt with the, descriptive or otherwise, set theoretic properties of the Einstein space $\mathbb{B}$. In order to discuss the dynamics of this space, and that of the P-sets, we find the following mathematical formalism useful.

Dynamical considerations

For an extended real-valued function $f : X \to \mathbb{R}$, the set $\{x \in X \mid f(x) \neq 0\} = \mathfrak{Support}(f)$ is a *support of* f *on* X. A function $f : X \to \mathbb{R}$ on a measurable space X is a *measurable function* if $\mathfrak{Support}(f) \bigcap f^{-1}(M)$ is a measurable set where M is any Borel subset of $\mathbb{R}$.

The forward image of a measurable subset of X under a measurable function f *need not be* measurable, in general.

Lusin's theorem: If f is a measurable function from a SBS into another SBS and if f is *countable to zero*, *ie*, if the inverse image of every singleton set is at most countable, then the forward image under f of a Borel set is Borel.

Now, a one-one measurable map T of a Borel space $(X, \mathcal{B})$ onto itself s.t. T^{-1} is also measurable is called a *Borel automorphism*. That is to say, a Borel automorphism of $(X, \mathcal{B})$ is a one-one and onto map $T : X \to X$ s.t. $T(B) \in \mathcal{B}\ \forall\ B \in \mathcal{B}$. We call a set A *invariant under* T or T*-invariant* if $TA = A$.

An automorphism of X onto X is, in general, not a Borel automorphism. But, if $(X, \mathcal{B})$ is a SBS then a measurable one-one map of X onto X is a Borel automorphism.

The Einstein-space $\mathbb{B}$ is a SBS. Hence, we will be considering only the Borel automorphisms of $(\mathbb{B}, \mathcal{B})$ and the class of Borel measures on it. This class will be that of physical attributes of the particle associated with a P-set.

Then, the Ramsay-Mackey theorem states that: If $T : X \to X$ is a Borel automorphism of a SBS $(X, \mathcal{B})$, then there exists a topology Γ on X s.t.

(a) (X, Γ) is a complete, separable, metric space

(b) Borel sets of (X, Γ) are precisely those in $\mathcal{B}$

(c) T is a homeomorphism of (X, Γ).

Note that X is *same* for $(X, \mathcal{B})$ and (X, Γ). However, this is not the usual formalism of the topological dynamics.

Alternatively, if X is the underlying set and if T is a Borel automorphism on a SBS $(X, \mathcal{B})$ and $\mathcal{C} \subseteq \mathcal{B}$ is a countable collection, then there exists a complete separable metric topology, *ie*, Polish topology, Γ, on X such that

(1) T generates the σ-algebra $\mathcal{B}$

(2) T is a homeomorphism of (X, Γ)

(3) $\mathcal{C} \subseteq \Gamma$, and lastly,

(4) Γ has a clopen base, *ie*, sets which are both open and closed in Γ.

If Γ_o is a Polish topology on X which generates $\mathcal{B}$, $\{\Gamma_i,\ i \in \mathbb{N}\}$ are Polish topologies on X with $\Gamma_o \subseteq \Gamma_i \subseteq \mathcal{B}$ then $\exists$ a Polish topology Γ_∞ $(\subseteq \mathcal{B})$ such that $\bigcup_{i=1}^{\infty} \Gamma_i \subseteq \Gamma_\infty$ and Γ_∞ is the topology generated by all *finite intersections* of the form $\bigcap_{i=1}^{n} G_i,\ G_i \in \Gamma_i$ for $i, n \in \mathbb{N}$.

Further, given $B \in \mathcal{B}$, there exists a Polish topology $\bar{\Gamma}$, $\Gamma_o \subseteq \bar{\Gamma} \subseteq \mathcal{B}$ s.t. $B \in \bar{\Gamma}$. Moreover, $\bar{\Gamma}$ can be chosen to have a clopen base.

Now, as noted, the Einstein space $\mathbb{B}$ is a Polish space with the topology generated by the Einstein pseudo-metric (2) being the Polish topology on $\mathbb{B}$. Also, every P-set is open in $\mathbb{B}$ but every open set of $\mathbb{B}$ is *not* a P-set.

However, it is clear that given any open subset $A \subset \mathbb{B}$, there is a Einstein pseudo-metric (2) s.t. the set A is a P-set, that is, the restriction of the Einstein pseudo-metric to $A \subset \mathbb{B}$ is a distance function. To achieve this, we select appropriate functions P, Q, R in (2) to make the subset A a P-set. Then, given $B \in \mathcal{B}$, there exists a suitable Einstein pseudo-metric (2) generating the topology $\bar{\Gamma}$ for which B is a open set.

That the Einstein pseudo-metric (2) can be defined on a SBS and the same pseudo-metric is a distance function on certain Borel subsets of a SBS, the P-sets, is a result of this paper.

We also note that the restriction of a Polish topology to a G_δ set is also a Polish topology. Furthermore, for any countable collection $(B_j)_{j=1}^{\infty} \subseteq \mathcal{B}$, there exists a Polish topology Γ (which can be chosen to have a clopen base) s.t. $\Gamma_o \subseteq \Gamma \subseteq \mathcal{B}$ and for all j, $B_j \in \Gamma$.

An extension of this result for *jointly measurable flows* is available in [11]. A group $T_t,\ t \in \mathbb{R}$ of Borel automorphisms on a SBS is a *jointly measurable flow* if

(1) the map $(T, x) \mapsto T_t x$ from $\mathbb{R} \times X \to X$ is measurable, where $\mathbb{R} \times X$ is endowed with the usual product Borel structure

(2) $T_0 x = x\ \forall\ x \in X$ and,

(3) $T_{t+s} x = T_t \circ T_s x$ for all $t, s \in \mathbb{R}$ and for all $x \in X$.

Further, if T is a homeomorphism of a Polish space X then there exists a compact metric space Y and a homeomorphism τ of Y s.t. T is isomorphic as a homeomorphism to the restriction of τ to a τ-invariant G_δ subset of Y. We can choose the τ-invariant set to be dense in Y. This result is due to N. Krylov and N. Bogoliouboff.

Combined with the theorem of Ramsay and Mackey, this shows that a Borel automorphism on a SBS can be viewed as a restriction of a homeomorphism of a compact metric space to an invariant G_δ subset.

Furthermore, in [17], it is proved that: Given a jointly measurable action of a second countable locally compact group G on a SBS $(X, \mathcal{B})$ there is a compact metric space Y on which G acts continuously and a Borel subset $X' \subseteq Y$ which is G-invariant and s.t. the G-actions on X' and X are isomorphic.

Moreover, Becker [7] has proved: If G is a Polish group acting continuously on a Polish space (X, Γ) and if $\mathcal{C}$ is a countable family of G-invariant Borel subsets of X, then there exists a Polish topology $\Gamma \supseteq \Gamma$ s.t.

(1) G acts continuously on (X, Γ_1)

(2) Γ_1 generates the same Borel structure $\mathcal{B}$ on X as Γ

(3) Every set in $\mathcal{C}$ is Γ_1 closed.

The generalization [7] of the Ramsay-Mackey theorem to locally compact second countable group actions (due to Kechris) states that: Let a second countable locally compact group G act in a jointly measurable fashion on a SBS $(X, \mathcal{B})$. Then, we can imbed X as a G-invariant Borel set in a compact metric space Y. We can then enlarge the topology of Y to be a Polish topology Γ_1 wrt which the G-action remains jointly continuous and s.t. X is closed under Γ_1. If we now restrict Γ_1 to X then X is Polish under this topology and the G-action on X is jointly continuous.

The Glimm-Effros Theorem [7] states that: If X is a complete separable metric space and G a group of homeomorphisms of X onto itself s.t. for some non-isolated point $x \in X$, the set Gx, the orbit of x under G, is dense in X, then there is a continuous probability measure μ on Borel subsets of X s.t. every G-invariant Borel set has measure zero or one.

Now, we say that a group G of homeomorphisms of a Polish space X admits a *recurrent point* x if there exists a sequence $(g_n)_{n=1}^{\infty}$ of elements in G s.t. $g_n x \neq x\ \forall\ n$ and $g_n x \to x$ as $n \to \infty$. A recurrent point x is not isolated in the closure of Gx and its G-orbit is clearly dense in the closure of Gx. It is easy to see that if the quotient topology on X/G is not T_o then the G-action on X admits a recurrent point.

A Borel set W is said to be *G-wandering* if the sets $gW,\ g \in G$ are pairwise disjoint. We write $\mathcal{W}_G$ for the σ-ideal generated by G-wandering Borel sets, it consists of *countable unions* of G-wandering sets of X.

Then, in this case, we have the result that: If a group of homeomorphisms G of a Polish space X acts *freely* and does not admit a recurrent point, then $X \in \mathcal{W}_G$.

That a P-set is *not* a singleton set of $\mathbb{B}$ is, of course, consistent with all such results.

Precisely, some of the above results guarantee the existence of P-sets on the base space $\mathbb{B}$. Also, certain theorems [7] establish the existence of a continuous finite measure on $\mathcal{B}$.

Now, since a SBS is Borel-isomorphic to the Borel space of the unit interval $X = [0, 1]$ equipped with the σ-algebra generated by its usual topology, we can restrict our discussion to it.

Then, for any $x \in X$, we define the *orbit of x under T* as the set $\{T^n x | n \in \mathbb{Z}\}$. We call a point $x \in X$ a *periodic point of X* if $T^n x = x$ for some integer n and call the smallest such integer the *period of x under T*.

For $A \subseteq X$ and $x \in A$, we say that the point x is *recurrent in A* if $T^n x \in A$ for infinitely many positive (fimp) n and for infinitely many negative (fimn) n and we call the point x a *recurrent point*. For a metric space (X, d), a point $x \in X$ is recurrent if $\liminf_{n\to\infty} d(x, T^n x) = 0$.

Two Borel automorphisms, T_1 on a Borel space $(X_1, \mathcal{B}_1)$ and T_2 on a Borel space $(X_2, \mathcal{B}_2)$, are said to be *isomorphic* if there exists a Borel isomorphism $\phi : X_1 \to X_2$ s.t. $\phi\, T_1\, \phi^{-1} = T_2$.

We also say that Borel automorphisms T_1 and T_2 are *weakly equivalent* or *orbit equivalent* if there exists a Borel automorphism $\phi : X_1 \to X_2$ s.t. $\phi\,(\mathrm{orb}(x, T_1)) = \mathrm{orb}\,(\phi(x), T_2),\ \forall\ \ x \in X_1$.

If two Borel automorphisms are isomorphic then they are also orbit-equivalent. However, the converse is, in general, not true.

Now, we say that a Borel automorphism T is an *elementary Borel automorphism* or that the *orbit space of T admits a Borel cross-section* or that *T admits a Borel cross-section* iff there exists a measurable set B which intersects each orbit under T in exactly one point.

Clearly, if n is the period of x under T, then the set $\{x, Tx, T^2x, ..., T^{n-1}x\}$ consists of *distinct* points of X. Now, for every positive integer n, let $E_n = \{x \,|\, Tx \neq x, ..., T^{n-1}x \neq x, T^n x = x\}$, and $E_\infty = \{x \,|\, T^n x \neq x \text{ for all integers } n\}$. Then, each $E_n,\ n < \infty$, is Borel, $E_m \bigcap E_n = \emptyset$ if $m \neq n$, and $\bigcup_{n=1}^{\infty} E_n = X$. Clearly, each E_n is a T-invariant Borel subset of X.

Now, if $y \in \{x, Tx, ..., T^{n-1}x\}$, then we clearly see that $\{x, Tx, ..., T^{n-1}x\} = \{y, Ty, ..., T^{n-1}y\}$. Moreover, due to the natural order on $[0, 1]$, if $y = \min\{x, Tx, ..., T^{n-1}x\}$, then $y < Ty$, $y < T^2y$, ..., $y < T^{n-1}y$, $y = T^n y$. Then, we can define $B_n = \{y \in E_n \,|\, y < Ty,\ ...,\ y < T^{n-1}y\}$.

Then, for $n < \infty$, B_n is a measurable subset of E_n and it contains exactly one point of the orbit of each $x \in E_n$. Note, however, that B_∞ need not be measurable.

Now, $X - E_\infty = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{n-1} T^k B_n$. The set $B = \bigcup_{k=1}^{\infty} B_k$ is Borel and has the property that orbit of any point in $X - E_\infty$ intersects B in exactly one point. Let $\mathbf{c}_n(T)$ denote the cardinality of B_n, $n < \infty$. The sequence of integers $\{\mathbf{c}_\infty(T), \mathbf{c}_1(T), \mathbf{c}_2(T), ...\}$ is called the *cardinality sequence associated to T*.

If T_1 and T_2 are orbit equivalent, then their associated cardinality sequences are the same. Also, if T_1 and T_2 are elementary and the associated cardinality sequences are the same, then T_1 and T_2 are isomorphic and orbit equivalent.

A measurable subset $W \subset X$ is *T-wandering* or *wandering under T* if $T^n W,\ n \in \mathbb{Z}$, are pairwise disjoint. Clearly, a wandering set intersects the orbit of any point in at most one point, it never intersects the orbit of a periodic point.

The σ-ideal generated by all T-wandering sets in $\mathcal{B}$ will be denoted by $\mathcal{W}_{_T}$ and will be called a Shelah-Weiss ideal of T [10].

Note that if T is a homeomorphism of a separable metric space (X, d) and T has no recurrent points then $\mathcal{W}_{_T} = \mathcal{B}$, *ie*, there is a wandering set W s.t. $X = \bigcup_{n=-\infty}^{\infty} T^n W$.

A subset $A \subset \mathrm{orb}(x, T)$ is called *bounded below* (*bounded above*) if the set of integers n s.t. $T^n x \in A$ is bounded below (bounded above). A subset $A \subset \mathrm{orb}(x, T)$ is called *bounded* iff it is both bounded above and below. A set which is not bounded is called *unbounded*.

A sufficient condition for a set $N \in \mathcal{B}$ to be a T-wandering set, *ie*, a sufficient condition for $N \in \mathbb{B}$ to belong to $\mathcal{W}_{_T}$, is that $\forall\ x \in X,\ N \bigcap \mathrm{orb}(x, T)$ is either bounded above or below.

One of the very basic results of the study of Borel automorphisms is:

Poincaré Recurrence Lemma: Let T be a Borel automorphism of a SBS $(X, \mathcal{B})$. Then, given $A \in \mathcal{B}$ $\exists\ N \in \mathcal{W}_{_T}$ s.t. $\forall\ x \in A_o = A - N$ the points $T^n x$ return to A fimp n and fimn n.

Now, note also that if $x \in A_o = A - N$ then $T^k x$ returns to A_o fimp k and fimn k because N is T-invariant and $x \notin N$.

Also, if $A \in \mathcal{B}$, and if $A_o = A - N$ is as in the Poincaré Recurrence Lemma, then $\bigcup_{k=-\infty}^{\infty} T^k A = \bigcup_{k=0}^{\infty} T^k A \ (\mathrm{mod}\ \mathcal{W}_{_T})$.

Now, suppose that $\mathcal{N} \subseteq \mathcal{B}$ is a σ-ideal s.t. $T\mathcal{N} = T^{-1}\mathcal{N} = \mathcal{N}$ and $\mathcal{W}_{_T} \subseteq \mathcal{N}$. Clearly, given $A \in \mathcal{B}$, $\exists\ N \in \mathcal{N}$ s.t. $\forall\ x \in A_o = A - N$, $T^n x$ returns to A_o fimp n and fimn n.

Of particular interest to us is a finite or σ-finite measure m on $\mathcal{B}$. The σ-ideal of m-null sets in $\mathcal{B}$ will be denoted by $\mathcal{N}_m$.

A Borel automorphism T is said to be *dissipative wrt m* if there exists a T-wandering set W in $\mathcal{B}$ s.t. m is supported on $\bigcup_{n=-\infty}^{\infty} T^n W$.

On the other hand, a Borel automorphism T is *conservative wrt* m or m*-conservative* if $m(W) = 0$ $\forall$ T-wandering sets $W \in \mathcal{B}$. Clearly, for any m-conservative T, $\mathcal{W}_T \subseteq \mathcal{N}_m$.

Poincaré Recurrence Lemma for m-conservative T: If T is m-conservative and if $A \in \mathcal{B}$ is given, then for almost every (f.a.e.) $x \in A$ the points $T^n x$ return to A fimp n and fimn n.

Further, if m is a probability measure on $\mathcal{B}$, *ie*, $m(X) = 1$, and is T-invariant, *ie*, $m \circ T^{-1} = m$, then $\mathcal{W}_T \subseteq \mathcal{N}_m$, $T\mathcal{N}_m = T^{-1}\mathcal{N}_m = \mathcal{N}_m$.

Poincaré Recurrence Lemma (measure theoretic): If a Borel automorphism T on $(X, \mathcal{B})$ preserves a probability measure on $\mathcal{B}$, and if $A \in \mathcal{B}$ is given, then f.a.e. $x \in A$ the points $T^n x$ return to A fimp n and fimn n.

Poincaré Recurrence Lemma (Baire Category): If T is a homeomorphism of a complete separable metric space X which has no T-wandering non-empty open set, then for every $A \subseteq X$ with the property of Baire (in particular, for any Borel set A) there exists a set N of the first Baire category (which is Borel if A is Borel) such that for each $x \in A - N$, the points $T^n x$ return to $A - N$ fimp n and fimn n.

Now, a measure-preserving automorphism T on a SPS $(X, \mathcal{B}, \mu)$ is a *Bernoulli-Shift* or *B-shift* if there exists a finite or a countably infinite partition $\mathcal{P} = \{P_1,\ P_2\ ...\}$ of X into measurable sets s.t.

(a) $\bigcup_{n=-\infty}^{\infty} T^n\mathcal{P}$ generates $\mathcal{B}_X$ up to μ-null sets

(b) the family $\{T^n\mathcal{P} \,|\, n \in \mathcal{Z}\}$ is independent in the sense that for all k, for all distinct integers n_1, n_2, ..., n_k, and for all P_1, P_2, ..., P_{i_k} $\in \mathcal{P}$, the sets $T^{n_1}P_{i_1}$, $T^{n_2}P_{i_2}$, ..., $T^{n_k}P_{i_k}$ are independent, *ie*, $\mu\left(T^{n_1}P_{i_1} \bigcap ... \bigcap T^{n_k}P_{i_k}\right) = \prod_{j=1}^{k} \mu\left(T^{n_j}P_{i_j}\right)$ which in view of the measure preserving character of T is equal to $\mu(P_{i_1})...\mu(P_{i_k})$.

We call the partition $\mathcal{P}$ satisfying the above an *independent generator of* T.

T is m*-deterministic* (otherwise, *non deterministic*) if $\forall$ n, $\mathcal{P}_n = \mathcal{P}_{n+1}$ (mod m) in that, given $A \in \mathcal{P}_n$, $\exists$ $B \in \mathcal{P}_{n+1}$ s.t. $m\,(A \triangle B) = 0$. If T is deterministic, $\mathcal{P}_n = \mathcal{P}_k \,(\text{mod}\, m)$, $\forall\, n, k$.

A non-deterministic Borel automorphism T is a *Kolmogorov Shift* or *K-shift* if $\bigcap_{n=-\infty}^{\infty} \mathcal{P}_n$ consists of sets with probability zero or one.

A B-shift is a K-shift and, hence, is of non-deterministic nature in the above sense.

Now, a measure preserving Borel automorphism T on a probability space $(X, \mathcal{B}, m)$ is said to be *ergodic* if for every T-invariant $A \in \mathcal{B}$, $m(A) = 0$ or $m(X - A) = 0$. Note that such a T is ergodic iff every real-valued measurable T-invariant function f is constant a.e.

Now, if T is measure-preserving, ergodic and for some singleton $\{x\} \in \mathcal{B}$, $m\,(\{x\}) > 0$, then x must be a periodic point of T. A non-trivial measure-preserving ergodic system is therefore the one for which m is non-atomic.

The system $(X, \mathcal{B}, \mathcal{N}, T)$ is called as a *descriptive dynamical system* [7].

Now, T is said to be *descriptively ergodic* or that T is said to *act in a descriptively ergodic manner* if $T\mathcal{N} = \mathcal{N}$ and if $TA = A,\ A \in \mathcal{B}$ implies either $A \in \mathcal{N}$ or $X - A \in \mathcal{N}$.

Nadkarni's theorem [7] states: if $(X, \mathcal{B}, \mathcal{N}, T)$ is a descriptive dynamical system s.t.

(a) every member of $\mathcal{B} - \mathcal{N}$ is decomposable

(b) $\mathcal{B}$ satisfies the countability condition

(c) T is descriptively ergodic

(d) X is bounded,

then there exists a finite measure μ on $\mathcal{B}$ s.t.

(1) $\mathcal{N} \ = \ \{B \in \mathcal{B} : \mu(B) = 0\}$

(2) μ is continuous

(3) T is μ-measure preserving, and

(4) T is ergodic, *ie*, $TA = A,\ A \in \mathcal{B}$ implies that $\mu(A) = 0$ or $\mu(X - A) = 0$.

As a corollary of this theorem, we also have: Let $(X, \mathcal{B}, \mathcal{N}, T)$ be a descriptive dynamical system s.t.

(a) every member of $\mathcal{B} - \mathcal{N}$ is decomposable,

(b) $\mathcal{B}$ satisfies the countability condition

(c) T is descriptively ergodic,

(d) $\exists$ $B \in \mathcal{B} - \mathcal{N}$ which is bounded

Then, there exists a unique continuous σ-finite measure m on $\mathcal{B}$ s.t. its null sets in $\mathcal{B}$ form precisely the ideal $\mathcal{N}$ and T is ergodic and measure preserving wrt m.

Furthermore, it can also be shown [7] that: for a SBS $(X, \mathcal{B})$ and $T : X \to X$ a Borel automorphism, there exists a finite continuous measure m on $\mathcal{B}$ so that T is *non-singular* and ergodic iff there exists a σ-ideal $\mathcal{N} \subseteq \mathcal{B}$ s.t. the system $(X, \mathcal{B}, \mathcal{N}, T)$ has the following properties

(i) every member of $\mathcal{B} - \mathcal{N}$ is decomposable

(ii) $\mathcal{B}$ satisfies the countability condition

(iii) T is descriptively ergodic

(iv) $\exists$ $B \in \mathcal{B} - \mathcal{N}$ which is bounded

Now, let $X = \{0,1\}^{\mathbb{N}}$, the countable product of two point space $\{0,1\}$ with product topology, with the two point space $\{0,1\}$ being given the discrete topology and $\mathcal{B}$ its Borel σ-algebra.

If we drop from above X the countable set of those sequences of zeros and ones which have only finitely many zeros or finitely many ones, then the remaining set, say, Y, can be mapped one-one into $[0,1)$ by the map $\xi(x_1,x_2,...) = \sum_{i=1}^{\infty} x_i/2^i$. The image of Y under this map is $[0,1) - D$ where D is the set of rational numbers of the form $k/2^n$, $0 \le k \le 2^n$, $n \in \mathbb{N}$.

Now, if $x = (x_1,x_2,...) \in Y$ and if k is the first integer s.t. $x_k = 0$, then let us define the map, say, $V = \xi^{-1}T\xi$ as $Vx = (0,0,...,0,1,x_{k+1},x_{k+2},...)$. Then, V replaces all the ones up to the first zero by zeros and replaces the first zero by one, leaving all other coordinates of x unchanged.

We call the map V on Y the *Diadic Adding Machine* (DAM) or the *Odometer*.

Now, a measure preserving automorphism T on a probability space $(X,\mathcal{B},m)$ is ergodic iff $\forall\ A,B \in \mathcal{B}$, $\frac{1}{n}\sum_{k=0}^{n-1} m\left(A \bigcap T^k B\right) \to m\left(A \bigcap B\right)$ as $n \to \infty$. There are two properties stronger than ergodicity which are also relevant to us.

A measure preserving automorphism T on a probability space $(X,\mathcal{B},m)$ is said to be *weakly mixing* iff $\forall\ A,B \in \mathcal{B}$, $\frac{1}{n}\sum_{k=0}^{n-1} \left| m\left(A \bigcap T^k B\right) - m\left(A \bigcap B\right) \right| \to 0$ as $n \to \infty$. A measuring preserving automorphism T on $(X,\mathcal{B},m)$ is said to be *mixing* if $\forall\ A,B \in \mathcal{B}$, $m\left(A \bigcap T^k B\right) \to m\left(A \bigcap B\right)$ as $n \to \infty$.

If a measure preserving Borel automorphism T is mixing then it is weakly mixing, and if T is weakly mixing then it is ergodic. However, a ergodic T need not be weakly mixing and mixing. Also, an ergodic and weakly mixing automorphism T need not be mixing.

Let T_1 be a measure preserving Borel automorphisms on a probability space $(X_1,\mathcal{B}_1,m_1)$ and T_2 be that on $(X_2,\mathcal{B}_2,m_2)$. We say that T_1 and T_2 are *metrically isomorphic* if $\exists\ X_1' \subseteq X_1$ with $m_1\left(X_1 - X_1'\right) = 0$, $X_2' \subseteq X_2$ with $m_2\left(X_2 - X_2'\right) = 0$ and an invertible, *ie*, a one-one, onto, measurable map with measurable inverse, measure preserving map $\phi : X_1' \to X_2'$ s.t. $\phi T_1 \phi^{-1} = T_2$.

A measure preserving automorphism T gives rise to a *Unitary Operator*, U_T, as: $U_T f = f \circ T$, $f \in L^2(X,\mathcal{B},m)$. The unitary operator is linear, invertible with $U_T^{-1} f = f \circ T^{-1}$ and L^2-norm preserving, *ie*, $||U_T f||_2 = ||f||_2$.

We say that λ is an *eigenvalue* of U_T if $\exists$ a non-zero $f \in L^2(X,\mathcal{B},m)$, s.t. $f \circ T = \lambda f$. Then,f an *eigenfunction* with eigenvalue λ. An eigenvalue is *simple*, if up to a multiplicative constant, it admits only one eigenfunction.

Let $L_o^2(X,\mathcal{B},m) = \{f \in L^2(X,\mathcal{B},m) \mid \int f dm = 0\}$, the subspace of functions orthogonal to the constant functions. It is U_T-invariant.

Now, 1 is always an eigenvalue of U_T and that 1 is a simple eigenvalue of U_T iff T is ergodic. Further, since U_T is unitary, all eigenvalues of U_T are of absolute value one.

Then, weakly mixing automorphisms T are precisely those for which U_T has no eigenvalue other than 1. Also, T is ergodic iff 1 is not an eigenvalue of U_T on $L_o^2(X,\mathcal{B},m)$.

If U_T and $U_{T'}$ are unitarily equivalent, T and T' are *spectrally isomorphic*. If measure preserving T and T' are metrically isomorphic, then U_T and $U_{T'}$ are unitarily equivalent.

A measure preserving automorphism T on a SPS $(X,\mathcal{B},m)$ has *discrete spectrum* if U_T admits a complete set of eigenfunctions. Then, if T_1 and T_2 are spectrally isomorphic and T_1 has a discrete spectrum, then T_2 also has a discrete spectrum and the corresponding unitary operators have the same set of eigenvalues.

But, spectrally isomorphic measure preserving automorphisms are not necessarily metrically isomorphic, in general. However, if the measure preserving automorphisms defined on a SPS are ergodic with discrete spectrum and are admitting the same set of eigenvalues, then such spectrally isomorphic measure preserving automorphisms are metrically isomorphic.

Note that in the case of a SPS, U_T can have at most a countable number of eigenvalues, all of absolute value one. Furthermore, in the same case, the eigenvalues of U_T form a subgroup of the circle group S^1. Also, for each eigenvalue λ we can choose an eigenfunction f_λ of absolute value one so as to have $f_\lambda . f_\nu = f_{\lambda\nu}$ a.e.

Any two B-shifts are spectrally isomorphic but, in general, any two B-shifts are not metrically isomorphic. Any two K-shifts are spectrally isomorphic but, in general, any two K-shifts are not metrically isomorphic.

For a finite partition $\mathcal{P} = \{P_1,P_2,...,P_k\}$ of X by members of $\mathcal{B}$, we define the *entropy* of $\mathcal{P}$ to be $\sum -\ m(P_i)\log_e m(P_i)$ and denote it by $H(\mathcal{P})$. Then, we have the *entropy of $\mathcal{P}$ relative to an automorphism T* defined as: $h(\mathcal{P},T) = \limsup \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} T^{-1}\mathcal{P}\right)$, where $\bigvee_{k=0}^{n-1} T^{-1}\mathcal{P}$ is used to denote the partition generated by $T^{-1}\mathcal{P}$, $k = 0,\ ...,\ n-1$. Note that the lim sup is indeed an increasing limit.

Then, we have the *entropy of the automorphism T*, denoted as $h(T)$, defined as: $h(T) = \sup\ h(\mathcal{P},T)$, where the supremum is taken over all finite partitions $\mathcal{P}$ of X. Note that $h(T)$ is an invariant of the metric isomorphism.

Then, if T is a B-shift with independent generating partition $\mathcal{P} = \{P_1, P_2, ...\}$ then its entropy is $h(T) = \sum - m(P_i) \log_e m(P_i)$. Now, any two B-shifts with the same entropy can be shown to be metrically isomorphic.

For any set $A \in \mathcal{B}$, the set $\bigcup_{k=-\infty}^{\infty} T^k A$ is called as the *saturation of A wrt T* or simply the *T-saturation of A*. We denote it by $s_T(A)$. A point $x \in A$ is said to be a *recurrent point in A* if $T^n x$ returns to A fimp n and fimn n.

By Poincaré Recurrence Lemma, we can write A as a disjoint union of two measurable sets B and M s.t. every point of B is recurrent in B (hence also in A) and no point of M is recurrent so that $M \in \mathcal{W}_T$. Clearly, it follows that $\bigcup_{n=0}^{\infty} T^n B = \bigcup_{n=-\infty}^{\infty} T^n B = s_T(B)$, since every point of B is recurrent in B.

At this point, we note that the P-sets partition the Einstein space $\mathbb{B}$, the partition being countably infinite. Further, each P-set is a measurable subset of $\mathbb{B}$ in $\mathcal{B}_{\mathbb{B}}$. Then, we will be dealing with measure preserving Borel automorphisms on the Einstein SPS $(\mathbb{B}, \mathcal{B}_{\mathbb{B}}, \mu)$ that are B-shifts.

Further, we also note that a P-set can be decomposed as a disjoint union of two measurable sets of $\mathcal{B}$ as per the above method based on the Poincaré Recurrence Lemma.

Now, given $x \in B$, let $n_B(x)$ denote the smallest positive integer s.t. $T^n x \in B$. Then, we can decompose B into pairwise disjoint sets $B_k,\ k \in \mathbb{N}$, where $B_k = \{x \in B \mid n_B(x) = k\}$ or, equivalently, $B_k = \{x \in B \mid Tx \notin B, ..., T^{k-1}x \notin B, T^k x \in B\}$. Further, we have $T^k B_k \subseteq B$ and that B_k, TB_k, ..., $T^{k-1}B_k$ are pairwise disjoint.

Further, let $F_\ell = T^\ell \left(\bigcup_{k>\ell} B_k\right)$ and note also that $F_\ell = TF_{\ell-1} - B$, where $F_o = B$. Now, we have $\bigcup_{k=0}^{\infty} T^k B = \bigcup_{k=0}^{\infty} \bigcup_{i=0}^{k-1} T^k B_k = \bigcup_{k=0}^{\infty} F_k = \bigcup_{k=-\infty}^{\infty} T^k B = s_T(B)$, with the middle two unions being pairwise disjoint unions.

We call the set B the *base* and the union $\bigcup_{k=1}^{\infty} T^{k-1} B_k$ the *top of the construction*. The above construction is called as the *Kakutani tower over base B*.

Now, if m is any T-invariant probability measure on $\mathcal{B}$ and if *we write* $B_\star = \bigcup_{k=0}^{\infty} T^k B$, then we have $m(B_\star) = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} m\left(T^i B_k\right) = \sum_{k=1}^{\infty} k\, m(B_k) = \int_B n_B(x)\, dm$.

Let $m(B) \geq 0$. Then, we call the quantity $\frac{1}{m(B)} \int_B n_B(x)\, dm = m(B_\star)/m(B)$ as the *mean recurrence time of B*. Recall $A = B \bigcup M$, $M \in \mathcal{W}_T$. Then, $m(M) = m(M_\star) = 0$. Hence, $m(A) = m(B)$ and $m(A_\star) = m(B_\star)$. Thus, the above is also the *mean recurrence time of A*.

Clearly, if T is ergodic and $m(B) > 0$ then we have $B_\star = X$ (mod m) since it is T-invariant and of positive measure.

Now, consider the transformation $\mathcal{S}$ defined over $B_\star$ as:

$$\mathcal{S}(x) = \begin{cases} T(x) & \text{if } x \notin \bigcup_{k=1}^{\infty} T^{k-1} B_k = \text{Top} \\ T^{-k+1}(x) & \text{if } x \in T^{k-1} B_k,\ k = 1, 2, ... \end{cases}$$

Then, $\mathcal{S}$ is periodic, the period being k for points in B_k, and $\mathcal{S}$ agrees with T everywhere except at the top of the Kakutani tower. Further, if $B_\star = X$, then $\mathcal{S}$ is defined on all of X.

Now, suppose $C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$ is a sequence of sets in $\mathcal{B}$ decreasing to an empty set and s.t. $\forall\ n$, we have

(i) every point of C_n is recurrent, and that

(ii) $\bigcup_{k=0}^{\infty} T^k C_n = X$.

Let $\mathcal{S}_n$ be the periodic automorphism as defined above with $B = C_n$. Then, $\forall\ n$, $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ agree except on the Top T_{n+1} of the Kakutani tower whose base is C_{n+1}. But $T_n \supseteq T_{n+1}$ and since C_n decreases to $\emptyset$, T_n also decreases to $\emptyset$. Then, given any x, $\exists\ n(x)$ s.t. $\forall\ k \geq n(x)$, $\mathcal{S}_k(x)$ are all the same and equal to $T(x)$. Thus, T is a limit in this sense of the sequence of periodic automorphisms. Hence, we obtain the *periodic approximation of automorphism T*.

A very useful result, Rokhlin's Lemma, states that: If T is ergodic wrt the σ-ideal of null sets of a finite measure m, then given $\epsilon > 0$ and $n \in \mathbb{N}$, $\exists$ a set C s.t. C, TC, ..., $T^{n-1}C$ are pairwise disjoint and $m\left(X - \bigcup_{k=o}^{n-1} T^k C\right) < \epsilon$.

Now, let $B \in \mathcal{B}$ be s.t. every point of B is recurrent. Following Kakutani, the *induced automorphism on B* (mod $\mathcal{W}_T$), denoted as T_B, is then defined as: $T_B(x) = T^n(x),\ x \in B$, where $n = n_B(x)$ is the smallest positive integer for which $T^n(x) \in B$. Note that $T_B(x) = T^k(x)$ if $x \in B_k$, $k = 1, 2, 3.....$ Then, T_B is one-one, measurable and invertible with $T^{-1}(x) = T^n(x)$ where n is the largest negative integer s.t. $T^n(x) \in B$. Thus, T_B is a Borel automorphism on B.

The induced Borel automorphism, T_B, on B has following properties:

- $\text{orb}(x, T_B) = B \bigcap \text{orb}(x, T)$, $x \in B$
- T_B is elementary iff T restricted to $s_T B$ is elementary
- $W \subseteq B$ is T_B-wandering iff W is T-wandering
- $\mathcal{W}_{T_B} = \mathcal{W}_T \bigcap B$
- if T is ergodic and preserving a finite measure m then T_B is ergodic and preserves m restricted to B,

- If $\mathcal{N}$ is a σ-ideal in $\mathcal{B}$, $\mathcal{W}_T \subseteq \mathcal{B}$, and if T is ergodic wrt $\mathcal{N}$, then T_B is ergodic wrt the restriction of $\mathcal{N}$ to B. In particular, if T is ergodic wrt a finite continuous measure m then T_B is ergodic wrt the restriction of m to B

- if $C \subseteq B$, then a point of C is recurrent wrt T iff it is recurrent wrt T_B. If every point of C is recurrent then we have $T_C = (T_B)_C$.

A broadened view of the induced automorphism defines it on a set $A \in \mathcal{B}$ even if not every point of A is recurrent. For this, let us consider a set $B = \{x \in A \mid x \text{ is recurrent in } A\}$. By Poincaré Recurrence Lemma, $A - B \in \mathcal{W}_T$ and every point of B is recurrent in B. Then, the broadened induced automorphism T_A is defined on all of A iff every point of A is recurrent; otherwise T_A is defined on A (mod $\mathcal{W}_T$). All the earlier properties of the induced automorphism remain valid (mod W_T) under this broadened definition of T_A. Note however that the stricter point of view is necessary for the descriptive aspects.

Now, consider a Borel automorphism T on $(X, \mathcal{B})$ and let f be a non-negative integer-valued measurable function on X.

Let $B_{k+1} = \{x \mid f(x) = k\}$, $k = 0, 1, 2, \ldots$, $C_k = \bigcup_{\ell > k} B_\ell$, $F_k = C_k \times \{k\}$, $Y = \bigcup_{k=0}^{\infty} F_k$. If $Z = X \times \{0, 1, 2, \ldots\}$, then $Y \subseteq Z$ is the set $Y = \{(x, n) \; 0 \geq n \geq f(x)\}$ = Points in Z below and including the graph of f.

Define Λ on Y as:

$$\Lambda(k, j) = \begin{cases} (b, j+1) & \text{if } b \in B_k \text{ and } 0 \leq j \leq k-1 \\ \\ (\Lambda(b), 0) & \text{if } b \in B_k \text{ and } j = k-1 \end{cases}$$

This Λ is a Borel automorphism on the space Y. We call it the *automorphism built under the function* f on the space X. We call X the *base space of* Λ and f the *ceiling function of* Λ. Note that if we identify X with $X \times \{0\}$, then $\Lambda_X = T$ and we write $\Lambda = T^f$.

The automorphism built under a function has the following properties:

- If $B \in \mathcal{B}$ with every point of B being recurrent and $B_\star = X$, then T is isomorphic to $(T_B)^f$, where $f(x) = n_B(x)$,

- If $A \subseteq Y$ is the graph of a measurable function ξ on X, then $(T^f)_A$ and T are isomorphic by $x \mapsto (x, \xi(x))$. In particular, $(T^f)_A$ and T are isomorphic when A = graph of f,

- If $A \subseteq Y$ is measurable then we can find a measurable B with the same saturation as A under T^f and s.t. $\forall\, x \in X$, $B \bigcap \{(x, i) \mid 0 \leq i \leq f(x)\}$ is at most a singleton. Indeed, $B = \{(x, i) \in A \mid (x, j) \notin A,\, 0 \leq j < i\}$ can be chosen,

- Given T^f and T^g, they are isomorphic to automorphisms induced by T^{f+g} on suitable subsets. If $Y_1 = \{(x, i) \mid 0 \leq i \leq f(x) + g(x)\}$ on which T^{f+g} is defined, then the sets $\{(x, i) \mid 0 \leq i \leq f(x)\}$ and $\{(x, i) \mid 0 \leq i \leq g(x)\}$ are subsets of Y_1 on which T^{f+g} induces automorphisms which are isomorphic to T^f and T^g respectively,

- If m is a σ-finite T-invariant measure on X, then $\exists$ a unique σ-finite T^f-invariant measure m_Y on Y s.t. m_Y restricted to $X \times \{0\}$ is m. The measure m_Y is finite iff $m(X)$ is finite and $\int f\, dm$ is finite. Then, we have $m_Y(Y) = \sum_{k=1}^{\infty} k\, m(B_{k+1}) = \int f\, dm < \infty$.

- T^f is elementary iff T is elementary.

Now, given two Borel automorphisms T_1 and T_2, we say that T_1 is a *derivative* of T_2, and write $T_1 \prec T_2$, if T_1 is isomorphic to $(T_1)_A$ for some $A \in \mathcal{B}$ with $\bigcup_{k=0}^{\infty} T_1^k A = X$. If T_1 is a derivative of T_2, we call T_2 the *primitive of* T_1. Two Borel automorphisms are said have a *common derivative* if they admit derivatives which are isomorphic. Similarly, two automorphisms are said to have a *common primitive* if they admit primitives which are isomorphic. If $T_1 \prec T_2$, then clearly $T_2 = T_1^f$ for some f.

Then, a lemma due to von Neumann states that: Two Borel automorphisms have a common derivative iff they have a common primitive.

Now, we say that two Borel automorphisms T_1 and T_2 are *Kakutani equivalent*, and we write $T_1 \underset{\mathrm{K}}{\sim} T_2$, if T_1 and T_2 have a common primitive, or, equivalently the automorphisms T_1 and T_2 have a common primitive. The Kakutani equivalence is reflexive, symmetric and transitive. Therefore, the Kakutani equivalence is an equivalence relation for Borel automorphisms.

Suppose $\mathcal{N}$ is a σ-ideal in $\mathcal{B}$. Then, we say that T_1 and T_2 are *Kakutani equivalent* (mod $\mathcal{N}$) if we can find two sets $M, N \in \mathcal{N}$, M being T_1-invariant and N being T_2-invariant, s.t. $T_1|_{X-M} \underset{\mathrm{K}}{\sim} T_2|_{X-N}$. When $\mathcal{N}$ is the σ-ideal of m-null sets of a probability measure m invariant under T_1 and T_2 both, we get the measure theoretic Kakutani equivalence of Borel automorphisms [12].

Given a Borel automorphism T, a system of pairwise disjoint sets $(C_o, C_1, \ldots, C_n) \in \mathcal{B}$ is called a *column* if $C_i = T^i C_o$, $0 \leq i \leq n$. C_o is called the *base of the column* and C_n is called the *top of the column.* If $D_o \subseteq C_o$, then $(D_o, TD_o, \ldots, T^n D_o)$ is called a *sub-column of* $(C_o, \ldots, C_n)$.

Two columns $(C_o, ..., C_n)$ and $(B_o, ..., B_m)$ are said to be *disjoint* if $C_i \bigcap B_j = \emptyset \; \forall \; i \neq j$. A finite or a countable system of pairwise disjoint columns is called a *T-tower*.

A T-tower with r pairwise distinct columns may be written as $\{C_{ij} \mid 0 \leq i \leq n(j), \; 1 \leq j \leq r\}$ where $\{C_{0j}, ..., C_{n(j)j}\}$ is its j-th column.

Sets C_{ij} are *constituents of the T-tower*, $\bigcup_k C_{0k}$ is a *base of the T-tower* and $\bigcup_k C_{n(k)k}$ is a *top of the T-tower*. The number of distinct columns in a T-tower is a *rank of the T-tower*.

A T-tower is said to *refine a S-tower* if every constituent of T-tower is a subset of a constituent of the S-tower.

T has *rank at most r* if there is a sequence $T_n, \; n \in \mathbb{N}$, of T_n-towers of rank r or less s.t. T_{n+1} refines T_n and the collection of sets in T_n, taken over all n, generates $\mathcal{B}$. Then, T has rank r if T has rank at most r but does not have rank at most $r-1$. If T does not have rank r for any finite r, then T has infinite rank.

Given a Borel automorphism T on $(X, \mathcal{B})$, a partition $\mathcal{P}$ of X, $\mathcal{P} \subseteq \mathcal{B}$, is a *generator of T* if $\bigcup_{k=1}^{\infty} T^k \mathcal{P}$ generates $\mathcal{B}$. A set $A \in \mathcal{B}$ is *decomposable* (mod $\mathcal{W}_T$) if we can write A as a disjoint union of two Borel sets C and D s.t. $s_T(C) = s_T(D) = s_T(A \; (\text{mod } \mathcal{W}_T))$.

Let $\mathcal{P} = \{P_1, P_2, ..., P_n\} \subseteq \mathcal{B}$ be a partition of X and let a measurable C be s.t. $\bigcup_{k=0}^{\infty} T^k C = X$. Then, on the basis of the first return time $n(x)$ of each $x \in C$ and pairwise disjoint sets $E_i = \{x \in C \mid n(x) = i\}$ with union $\bigcup_i E_i = C$, there exists a countable partition of $\{D_1, D_2, ...\}$ of C s.t. each P_i is a disjoint union of sets of the form $T^k D_i$, $k = 1, 2, ..., \; i = 1, 2, ...$

Now, a one-one and onto map $T : X \to X$ s.t. $T^k x \neq x$ for all $k \neq 0$, and for all $x \in X$ is called a *free map*.

Every free Borel automorphism T on a SBS $(X, \mathcal{B})$ is [13] orbit equivalent to an induced automorphism by the DAM.

Further, every Borel set $A \in \mathcal{B}$ is clearly decomposable (mod $\mathcal{W}_T$) for T being a free Borel automorphism on a countably generated and countably separated SBS.

Furthermore, given a free Borel automorphism T on a countably generated and countably separated SBS $(X, \mathcal{B})$, there exists a sequence $C_n, \; n \in \mathbb{N}$, of Borel sets decreasing to an empty set with $s_T(C_n) = s_T(X - C_n) = X \; \forall \; n$, s.t. $\forall \; n$ the sets $C_n, TC_n, ..., T^{n-1}C_n$ are pairwise disjoint, and s.t. $\bigcap_{n=1}^{\infty} C_n = C_\infty$, say, is T-wandering.

Also, given a Borel automorphism T on a countably generated and countably separated SBS $(X, \mathcal{B})$, there exists a sequence $T_n, \; n = 1, 2, ...$ of periodic Borel automorphisms on X s.t. $\forall \; x$, $Tx = T_n x$ for all sufficiently large n.

Hence, the descriptive version of Rokhlin's theorem [14] on generators is obtained [15] as: every free Borel automorphism on a countably generated and countably separated SBS admits a countable generator in a strict sense.

Note also that T admits a countable generator iff T admits at most a countable number of periodic points [16].

Now, two subsets of X, $A, B \in \mathcal{B}$, are said to be *equivalent by countable decomposition*, and we write $A \sim B$, if

(a) $A = \bigcup_{i=1}^{\infty} A_i, \; A_i \bigcap A_j = \emptyset$ for $i \neq j$, and $A_i \in \mathcal{B}, \; i = 1, 2, ...$

(b) $B = \bigcup_{i=1}^{\infty} B_i, \; B_i \bigcap B_j = \emptyset$ for $i \neq j$, and $B_i \in \mathcal{B}, \; i = 1, 2, ...$

(c) there exist $n_1, \; n_2, \; ... \in \mathbb{N}$ s.t. $\forall \; i \in \mathbb{N}, \; T^{n_i} A_i = B_i \; (\text{mod} \, \mathcal{N})$.

The equivalence by countable decomposition is an equivalence relation on $\mathcal{B}$.

Note that if $A_i \in \mathcal{B}, \; i \in \mathbb{N}$ are pairwise disjoint and $B_i \in \mathcal{B}, \; i \in \mathbb{N}$ are pairwise disjoint and if $\forall \; i \in \mathbb{N}, \; A_i \sim B_i$ then $\bigcup_{i=1}^{\infty} A_i \; \sim \; \bigcup_{i=1}^{\infty} B_i$.

If $A \sim B$, then we say that B is a *copy of A* and then, A and B have the same measure wrt a T-invariant σ-finite measure.

Further, we say that A and B are *equivalent by countable decomposition* (mod m), and we write $A \sim B$ (mod m), if there exist sets M and N in $\mathcal{B}$, of m-measure zero, s.t. $A \triangle M \sim B \triangle N$.

A set $A \in \mathcal{B}$ is said to be *T-compressible in the sense of Hopf* if there exists $B \subsetneq A$ s.t $A \sim B$ and $m(A - B) > 0$. Clearly, if the set X is Hopf T-compressible then every of its subsets $B \in \mathcal{B}$ is Hopf T-compressible.

If μ is a T-invariant finite measure on $\mathcal{B}$ and having the same null sets as m, then $A \sim B$ (mod m) implies that $\mu(A) = \mu(B)$. Whenever such a μ exists, no measurable sets of positive measure can be compressible in the sense of Hopf and, in particular, X is not Hopf T-compressible.

In a descriptive setting, one can dispense with the measure and consider only a SBS $(X, \mathcal{B})$ and a free Borel automorphism T on it.

Then, given $A, B \in \mathcal{B}$, we write $A \prec\prec B$ if there exists a measurable subset $C \subseteq B$ s.t. $A \sim C$ and $s_T(B - C) = s_T B$, which is the smallest T-invariant set containing B.

Now, we say that A is *T-compressible* if $A \prec\prec A$ or, equivalently, if we can write A as a disjoint union of two sets $C, D \in \mathcal{B}$ s.t. $A \sim C$, and $s_T(A) = s_T(C) = s_T(D)$. The sets C and D together with the automorphism T which accomplishes $A \sim C$ is called a *compression of A*.

If X is T-compressible, we say that T *is compressible* or that T *compresses* X.

The above notion of compressibility has the following properties:

- If $A \in \mathcal{B}$ is T-compressible then any superset of A in $\mathcal{B}$ having the same saturation as A is compressible. In particular, $s_T(A)$ is T-compressible whenever A is T-compressible,
- Since T is a free automorphism, each orbit is infinite and T-compressible as also the saturation of any T-wandering set. However, every T-compressible T-invariant set in $\mathcal{B}$ is *not* the saturation of a T-wandering set in $\mathcal{B}$ except in special cases,
- A finite non-empty set is not T-compressible nor is a set A T-compressible if the orbit of some point intersects A in a finite non-empty set. Further, if there exists a T-invariant probability measure on $\mathcal{B}$, then no set of positive measure is T-compressible. In particular, X is not T-compressible in this case,
- Clearly, a subset of a T-compressible set need not be T-compressible,
- If $E \in \mathcal{B}$ is T-invariant, T-compressible, and if $F \in \mathcal{B}$ is a T-invariant subset of E, then F is T-compressible. The countable pairwise disjoint union of T-invariant, T-compressible sets in $\mathcal{B}$ is T-compressible. Clearly, any countable union of T-invariant, T-compressible sets in $\mathcal{B}$ is T-compressible,
- T-compressible sets in $\mathcal{B}$ do not form a σ-ideal in $\mathcal{B}$. However, T-invariant, T-compressible sets in $\mathcal{B}$ are closed under countable union and taking of T-invariant subsets in $\mathcal{B}$. Hence, the collection $\mathcal{H}$ of subsets in $\mathcal{B}$ whose saturations are T-compressible forms a σ-ideal in $\mathcal{B}$ and we call $\mathcal{H}$ the *Hopf ideal*.
- $\mathcal{W}_T = \mathcal{H}$ iff $X \in \mathcal{W}_T$.

Note that the Hopf ideal is also equal to the σ-ideal generated by T-compressible sets in $\mathcal{B}$. Note that $\mathcal{W}_T \subseteq \mathcal{H}$ since the saturation of every T-wandering set in $\mathcal{W}_T$ is T-compressible,

Let $\mathcal{N} \subseteq \mathcal{B}$ be a σ-ideal s.t.

(1) $T\mathcal{N} = T^{-1}\mathcal{N} = \mathcal{N}$ and

(2) $\mathcal{W}_T \subseteq \mathcal{N}$.

The Hopf ideal $\mathcal{H}$; the σ-ideal of m-null sets in $\mathcal{B}$ for any T-invariant σ-finite measure on $\mathcal{B}$; and the σ-ideal of m-mull sets when T is m-conservative are few such ideals.

Then, two sets A, $B \in \mathcal{B}$ are said to be *equivalent by countable decomposition* (mod $\mathcal{N}$) if we can find sets M, $N \in \mathcal{N}$ s.t. $A \triangle M \sim B \triangle N$. We then write $A \sim B$ (mod $\mathcal{N}$). Note that if $A \sim B$ (mod $\mathcal{N}$) then $s_T(A) = s_T(B)$ (mod $\mathcal{N}$). We write $A \prec\prec B$ (mod $\mathcal{N}$) if there exists a set $N \in \mathcal{N}$ s.t. $A \triangle N \prec\prec B \triangle N$.

A set A is *compressible* (mod $\mathcal{N}$) if $\exists\, N \in \mathcal{N}$ s.t. $A \triangle N$ is T-compressible. For a T-invariant set in $\mathcal{B}$ all the three notions of compressibility, namely, T-compressibility, compressibility (mod $\mathcal{W}_T$) and compressibility (mod $\mathcal{H}$), are equivalent.

Now, suppose that $A, B \in \mathcal{B}$ are equivalent by countable decomposition. Let $A = \bigcup_{i=1}^{\infty} A_i$, $B = \bigcup_{i=1}^{\infty} B_i$ be pairwise disjoint partitions of A and B respectively, s.t. for suitable integers n_i, $i \in \mathbb{N}$, $T^{n_i} A_i = B_i$.

The map $S : A \to B$ defined by $S(x) = T^{n_i}x$ if $x \in A_i$ is an *orbit preserving isomorphism* between A and B. In case A and B are equivalent by countable decomposition (mod $\mathcal{N}$) then S will be defined between $A \triangle N$ and $B \triangle M$ for suitable sets $M, n \in \mathcal{N}$. Such a S is an *orbit preserving isomorphism* between A and B (mod $\mathcal{N}$).

The following results are then easily obtainable for $A, B, C, D \in \mathcal{B}$:

(a) If $A \supseteq B \supseteq C$ and $A \sim C$ then $A \sim B$

(b) If $A \sim C \subseteq B$ and $B \sim D \subseteq A$ then $A \sim B$,

(c) If $A \supseteq B \supseteq C$ (mod $\mathcal{N}$) and $A \sim C$ (mod $\mathcal{N}$), then $A \sim B$ (mod $\mathcal{N}$),

(d) If $A \sim C$ (mod $\mathcal{N}$), $C \subseteq B$ (mod $\mathcal{N}$), and $B \sim D$ (mod $\mathcal{N}$), $D \subseteq A$ (mod $\mathcal{N}$), then $A \sim B$ (mod $\mathcal{N}$).

Note that for (c) and (d) we remove suitable sets in $\mathcal{N}$ from A, B, C, D.

Now, a set $A \in \mathcal{B}$ is *incompressible* if it is not compressible and it is *incompressible* (mod $\mathcal{N}$) if it is not compressible (mod $\mathcal{N}$). Note however that $A \in \mathcal{B}$ is incompressible (mod $\mathcal{N}$) does not mean that $A \triangle N$ is incompressible for a suitable set $N \in \mathcal{N}$. Note also that for a set in $\mathcal{B}$ to be incompressible (mod $\mathcal{N}$) it is sufficient that its saturation is incompressible (mod $\mathcal{N}$).

Let N be a positive integer. Then, it is easy to see that there exists $B \in \mathcal{B}$ s.t. $s_T(B) = X$ and $\forall\, x \in B$, its first return time, $n_B(x)$, is s.t. $N \leq n_B(x) \leq 2N$.

For any $F \in \mathcal{B}$ and $x \in X$, let us now define $r_\star(x, F) = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_F(T^k x)$ and $r^\star(x, F) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_F(T^k x)$ where $\mathbf{1}_F$ is the identity function on set F.

Then, we note that given $0 \leq b \leq 1$ and $\epsilon > 0$, there exists $F \in \mathcal{B}$ s.t. $s_T(F) = X$ and $b - \epsilon < r_\star(x, F)$, $r^\star(x, F) < b + \epsilon$.

We also note that, if $0 < b < 1$, then there exists $F \in \mathcal{B}$ s.t. $s_{_T}(F) = s_{_T}(X - F)$ and for all $x \in X$, $0 < r^\star(x, F) < b$.

Further, for any $F \in \mathcal{B}$ and $\epsilon > 0$, there exists a measurable $G \subseteq F$ s.t. $s_{_T}(G) = s_{_T}(F - G) = s_{_T}(F)$ and $r^\star(x, G) < \epsilon \pmod{\mathcal{W}_{_T}}$.

Now, we have a key dichotomy: Let $E, F \in \mathcal{B}$ and let $f = \mathbf{1}_{_E} - \mathbf{1}_{_F}$. Then, there exists a T-invariant set $N \in \mathcal{W}_{_T}$ s.t. if $x \in X - N$, then either

(a) $\forall\ y \in\ $ orb(x, T), there exists $n \geq 0$ s.t. $\sum_{k=0}^{n} f(T^k y) \geq 0$,

Or

(b) the set of $y \in\ $ orb(x, T) s.t. $\forall\ n \geq 0$, $\sum_{k=0}^{n} f(T^k y) < 0$ is unbounded to the left and right.

These are *mutually exclusive* conditions.

Furthermore, consider any decomposition of X into pairwise disjoint T-invariant sets X_o, X_1 X_2, N with $N \in \mathcal{N}$ and X_o, X_1 X_2 satisfying the properties

(c) $E \bigcap X_1 \prec\prec F \bigcap X_1$,

(d) $E \bigcap X_o\ \sim\ F \bigcap X_o$,

(e) $E \bigcap X_2 \prec\prec F \bigcap X_2$.

Such a decomposition will have the properties that

- for $x \in X_1 \pmod{\mathcal{H}}$ the set, say, $A(x) = \{y \in\ \text{orb}(x, T) \mid \sum_{k=0}^{n} f(T^n y) > 0\ \forall\ n \geq 0\}$ is unbounded to left and right,
- for any $x \in X_o \pmod{\mathcal{H}}$, for all $y \in$ orb(x, T) $\exists\, n \geq 0$ s.t. $\sum_{k=0}^{n} f(T^k y) = 0$,
- for $x \in X_2 (\text{mod } \mathcal{H})$, the set, say, $B(x) = \{y \in\ \text{orb}(x, T) \mid \sum_{k=0}^{n} f(T^n y) < 0\ \forall\ n \geq 0\}$ is unbounded to left and right.

Moreover, (mod $\mathcal{H}$), we have that

$$\begin{aligned}
\{x \mid r_\star(x, E)\ <\ r_\star(x, F)\} &\ \subseteq\ X_2,\\
\{x \mid r^\star(x, E)\ <\ r^\star(x, F)\} &\ \subseteq\ X_2,\\
\{x \mid r_\star(x, E)\ >\ r_\star(x, F)\} &\ \subseteq\ X_1,\\
\{x \mid r^\star(x, E)\ >\ r^\star(x, F)\} &\ \subseteq\ X_1
\end{aligned}$$

Then, we have the following measure free version of the Birkhoff point-wise Ergodic Theorem as: For any $E \in \mathcal{B}$, the set of points x for which limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_E(T^k x)$ does not exist belongs to the Hopf ideal $\mathcal{H}$. That is to say, the set $\{x \mid r_\star(x, E)\ <\ r^\star(x, E)\}$ is compressible.

For any $E \in \mathcal{B}$, let us now write $m(E, x) = \lim \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{_E}(T^k x)$. This m is countably additive (mod $\mathcal{H}$) and T-invariant. Moreover, we can show that $m(E, x) = 0 \pmod{\mathcal{H}}$ iff $E \in \mathcal{H}$.

Now, let the Polish topology Γ on X possess a countable clopen base $\mathcal{U}$ that is closed under complements, finite unions and applications of T. There then exists a T-invariant set $N \in \mathcal{H}$ s.t. $\forall\ x \in X - N$, $m(A \bigcup B, x) = m(A, x) + m(B, x)$ whenever $A, B \in \mathcal{U}$ and $A \bigcap B = \emptyset$.

Fix $x \in X - N$ and let us write $m(A, x) = m(A)$, $A \in \mathcal{U}$. For any $B \subseteq X$, let us define $m^\star(B) = \inf\{\sum_{i=1}^{\infty} m(U_i) \mid B \subseteq \bigcup_{i=1}^{\infty} U_i\,,\ U_i \in \mathcal{U}\ \forall\, i\}$. This $m^\star$, an outer measure on $\mathfrak{P}(X)$, is T-invariant, bounded by one and $m^\star(X) = 1$.

Recall [9] that an outer measure $\mu^\star$ on the power set of a metric space (X, d) is called a *metric outer measure* if $\mu^\star(E \bigcup F) = \mu^\star(E) + \mu^\star(F)$ whenever $d(E, F) > 0$. If $\mu^\star$ is a metric outer measure on (X, d) then all open sets, hence, all Borel sets, are $\mu^\star$-measurable. Then, $m^\star$ defined above is a metric outer measure on X. The restriction of $m^\star$ to $\mathcal{B}$ is a countably additive T-invariant probability measure on $\mathcal{B}$.

Further, if T is not free, then it has a periodic point on whose orbit we can always put a T-invariant probability measure.

Therefore, if T is a Borel automorphism (free or not) of a SBS $(S, \mathcal{B})$ s.t. X is T-incompressible, then there exists a T-invariant probability measure on $\mathcal{B}$. This is Hopf's theorem.

Now, a set $A \in \mathcal{B}$ is *weakly T-wandering* if $T^n A$ are pairwise disjoint for n in some infinite subset of integers. Then, a non-singular automorphism T on a probability space $(X, \mathcal{B}, m)$ admits [18] an equivalent T-invariant probability measure iff there is no weakly T-wandering set of positive measure.

But, T-compressibility of X does not imply the existence of a weakly T-wandering set $W \in \mathcal{B}$ s.t. $s_{_T}(W) = X$ [19]. If a measurable $A \in \mathcal{B}$ is T-compressible then $s_{_T}(A) \prec\prec A$ and $s_{_T} \sim A$.

Let T_1 and T_2 be Borel automorphisms on a SBS. Then, if T_1 and T_2 are orbit equivalent and if T_1 has an orbit of length n then so has T_2 and vice versa. Moreover, the cardinality of the set of orbits of length n for T_1 and T_2 is the same. Further, if $\mathbf{c}_k(T_1)$ is the cardinality of the set of orbits of length k, then for each $k \leq \aleph_o$, $\mathbf{c}_k(T_1) = \mathbf{c}_k(T_2)$ whenever T_1 and T_2 are orbit equivalent.

Dye's theorem [20] proves that: any two free ergodic measure preserving Borel automorphisms on a SPS $(X, \mathcal{B}, m)$ are orbit-equivalent (mod m). Furthermore, we also note that if T_1 and T_2 are Borel automorphisms both compressible and not admitting Borel cross-sections, then T_1 and T_2 are orbit-equivalent [21].

Let $M(X) = M(X, \mathcal{B}, m)$ be the group of all measure preserving automorphisms on the space $(X, \mathcal{B}, m)$. Two automorphisms in M are identified if they agree a.e.

For a $T \in M$, let us denote by $[T]$, called the *full group of* T, the collection of all $\tau \in M$ s.t. f.a.e. $x \in X$, $\tau(x) = T^n(x)$ for some integer $n = n(x)$. Note that $\tau \in [T]$ iff $\mathrm{orb}(x, \tau) \subseteq \mathrm{orb}(x, T)$ f.a.e $x \in X$, or equivalently, there exists a decomposition of $X = \bigcup_{n \in \mathbb{Z}} A_n \pmod m$ s.t. $X = \bigcup_{n \in \mathbb{Z}} T^n A_n \pmod m$, $T^n A_n$ being pairwise disjoint, and $\tau(x) = T^n(x)$ for $x \in A_n$, $n \in \mathbb{Z}$.

Let $A \in \mathcal{B}$ and $\tau \in [T]$. We shall write $\tau \in [T]^+$ on A in case $\tau(x) = T^n(x)$, where $n = n(x) > 0$ a.e. on A.

An automorphism T is called *set periodic with period* k, for some positive integer k, if there exists a partition $\mathcal{P} = \{D_1, D_2, ..., D_k\}$ of X associated with T s.t. $D_i = T^{i-1} D_1$, for $1 \le i \le k$ with each $D_i \in \mathcal{B}$.

If every x is T-periodic with period k, then it is clear that T is set periodic with period k. However, it should also be noted that T can be set periodic without having any periodic points.

An automorphism $T \in M(X)$ is called a *weak von Neumann automorphism* if

(1) T is set periodic with period 2^n for every $n \in \mathbb{N}$,

(2) There exists a sequence $\{\mathcal{D}_n(T) = (D_1^n, ..., D_{2^n}^n)\}$, $n \in \mathbb{N}$, of partitions of X associated with T satisfying

  (a) $D_i^n = D_i^{n+1} \bigcup D_{i+2^n}^{n+1}$, for $i = 1, 2, ..., 2^n$, $n \in \mathbb{N}$

  (b) $D_i^n = T^{i-1} D_i^n$, for $i = 1, 2, ..., 2^n$, $n \in \mathbb{N}$.

For $x \in D_1^n$, we call the finite sequence $(x, Tx, ..., T^{2^n - 1}x)$ a *fiber of length* 2^n. Two points $u, v \in X$ are said to be *in the same fiber of length* 2^n if for some $x \in D_1^n$, $u = T^k x$, $v = T^\ell x$, where $0 \le k, \ell < 2^n - 1$.

If, in addition to the above (1) and (2), we have

(3) the σ-field generated by $\bigcup_{n=1}^\infty \mathcal{D}_n(T)$ is equal to $\mathcal{B} \pmod m$,

T is called as a *von Neumann automorphism.*

This above condition (c) means that there exists a T-invariant set $N \in \mathcal{B}$ which is m-null and s.t. the collection $\{D \bigcap (X - n) \mid D \in \bigcup_{n=1}^\infty \mathcal{D}_n(T)\}$ generates the σ-algebra $\mathcal{B}$ restricted to $X - N$, equivalently, the sets D_k^n taken over all n and all k separate the points of $X - N$.

For a weak von Neumann automorphism T, let $\mathcal{P}_n(T)$ denote the algebra generated by $\mathcal{D}_n(T)$. Then, $\mathcal{P}_n(T) \subseteq \mathcal{P}_{n+1}(T)$ and the union $\mathcal{P}(T) = \bigcup_{n=1}^\infty \mathcal{P}_n(T)$ is again an algebra. For $A \in \mathcal{B}$, write $d(A) = \inf\{m(A \triangle B) \mid B \in \mathcal{P}(T)\}$. If $d(A) = 0$ for every A in a countable collection which generates $\mathcal{B}$ then T is a von Neumann automorphism.

A DAM or Odometer V on $\{0,1\}^{\mathbb{N}}$ is a von Neumann automorphism. Furthermore, any two von Neumann automorphisms are isomorphic modulo m-null sets.

Now, for ergodic $T \in M(X)$ and $\forall\ A, B \in \mathcal{B}$ with $0 < m(A) = m(B)$, there exists a $J \in [T]$ s.t. $JB = A$ and $J \in [T]^+$ on B. Therefore, if $m(A) = m(B)$, then T_A and T_B are orbit equivalent. Indeed, J when viewed as an isomorphism from A to B establishes orbit equivalence $\pmod m$ between T_A and T_B.

Moreover, let $T \in M(X)$ be ergodic and let $\epsilon > 0$ be s.t. $\epsilon < m(X)$. Then, there exists $A \in \mathcal{B}$ s.t. $A \bigcap TA = \emptyset$ and $m(X - A \bigcup TA) = \epsilon$. Also, there exists a weak von Neumann automorphism $\omega \in [T]$ s.t. $[\omega] = [T]$.

Further, if $\tau_1 \in [T]$ is a set periodic automorphism with period 2^K s.t. $\mathcal{D}(\tau_1) = (D_1, ..., D_{2^K})$ is a partition of X associated with τ_1, then, for any $\epsilon > 0$ and any set $A \in \mathcal{B}$, there exists a weak von Neumann automorphism $\tau_1 \in [T]$ and an integer $L > 0$ that satisfy

(a) $[\tau_1] = [\tau_2]$

(b) $\mathcal{D}(\tau_1) \subseteq \mathcal{D}_n(\tau_2)$ for all $n \ge L$, where $\{\mathcal{D}_n(\tau_2) \mid n \in \mathbb{N}\}$ are the partitions of X associated with τ_2

(c) $\{x \mid \tau_2(x) \ne \tau_1(x)\} \subseteq D_{2^K} \in \mathcal{D}(\tau_1)$

(d) for $n \ge L$, we have $m(A - A'_n) < \epsilon$, $m(A'' - A) < \epsilon$, where $A'_n = \bigcup D$ where union is over $\mathcal{D}'_n = \{D \in \mathcal{D}_n(\tau_2) \mid D \subseteq A\}$ and $A''_n = \bigcup D$ where the union is over $\mathcal{D}''_n = \{D \in \mathcal{D}_n(\tau_2) \mid m(A \bigcap D) > 0\}$.

Under the same hypotheses as above, if we have in addition that $\tau_1 \in [T]^+$ on $X - D_{2^K}$ for $D_{2^K} \in \mathcal{D}(\tau_1)$, then the weak von Neumann automorphism $\tau_2 \in [T]$ and the positive integer $L > 0$ chosen above also satisfy

(e) $\tau_2 \in [T]^+$ on $X - D_{2^L}^L$ for $D_{2^L}^L \in \mathcal{D}_L(\tau_2)$.

Furthermore, there exists an integer $P > L$, and $C \in \mathcal{B}$ with $m(C) < \epsilon$ s.t. the following holds:

(f) $C(x, Tx)$ does not intersect $D_{2^P}^P \in \mathcal{D}_P$ for all $x \in X - C$, where for $y \in \mathrm{orb}(x, \tau_2)$ with $\tau_2^{n(x)} x = y$ and $C(x, y) = (x, \tau_2 x, ..., \tau_2^n x = y)$, if $n = n(x) \ge 0$ and $C(x, y) = (x, \tau_2^{-1} x, ..., \tau_2^n x = y)$, if $n = n(x) < 0$.

In other words, x and Tx belong to the same τ_2-fiber of length 2^P for any $x \in X - C$.

Then, given a free ergodic measure preserving automorphism T on a SPS $(X, \mathcal{B}, m)$, there exist two von Neumann automorphisms τ_1 and τ_2 in $[T]$ s.t. (i) $\tau_1 \in [T]^+$ on X and (ii) $[\tau_1] = [\tau_2]$.

Note that when two Borel automorphisms on $(X, \mathcal{B})$ are free and uniquely ergodic, then the orbit equivalence holds without discarding any set of measure zero. Moreover, any two free Borel automorphisms on $(X, \mathcal{B})$, each admitting n invariant ergodic probability measures, are orbit equivalent whether we have n as finite or countable or uncountable [13].

Now, we note that Krieger [22] introduces an invariant called the *ratio set, $r(T)$, of automorphism* T as a closed subset of $[0, \infty)$ and $r(T) \bigcap (0, \infty)$ is a closed multiplicative subgroup of $(0, \infty)$. Then, if $r(T) = r(\tau) = [0, \infty)$ or if $r(T) = r(\tau) = \{0\} \bigcup \{\alpha^k \mid k \in \mathbb{Z}\}$ for some $\alpha,\ 0 < \alpha < 1$, then T and τ are orbit equivalent (mod m).

Extending these concepts to more general group actions is possible. Then, let G be Polish group of Borel automorphisms acting in a jointly measurable manner on a SBS $(X, \mathcal{B})$. Then, if X is incompressible wrt the G-action then there exists a probability measure on $\mathcal{B}$ invariant under the G-action [7].

However, note that further generalizations than above are limited by counter examples.

For example, let G now denote the group of *all* Borel automorphisms of an uncountable Polish space with the property that the set $\{x \mid gx \neq x\}$ is of the first Baire category. Then, X is not compressible. (We note that the Einstein space is of this type.)

A set of the first Baire category is also called as a *meagre set*. Now, the σ-ideal $\mathcal{H}_G$ generated by G-compressible sets in $\mathcal{B}$ is the σ-ideal of meagre Borel subsets of X. Hence, $X \notin \mathcal{H}_G$. However, every probability measure on $\mathcal{B}$ is supported on a meagre set. Therefore, a G-invariant probability measure on $\mathcal{B}$ does not exist.

A *flow* on a SBS $(X, \mathcal{B})$ is said to be *non-singular* wrt a σ-finite measure μ on $\mathcal{B}$ if $\mu(A) = 0$ implies that $\mu(T_t(A)) = 0$ for all $A \in \mathcal{B}$ and $t \in \mathbb{R}$. In case, $\mu(T_t(A)) = \mu(A)$ for all $t \in \mathbb{R}$ and $A \in \mathcal{B}$, then we say that the *flow preserves* μ.

Consider topology Γ on X in place of the Borel structure $\mathcal{B}$. If the map $(t, x) \mapsto T_t x$ is continuous on $\mathbb{R} \times X$, then the flow is *jointly continuous* or simply *continuous*. Here $\mathbb{R}$ is given the usual topology and $\mathbb{R} \times X$ is given the product topology. Note that each T_t is a homeomorphism of X onto itself. We also call a jointly continuous flow a *flow of homeomorphisms*. It is often called as a *one parameter group of automorphisms*.

Let σ be a Borel automorphism on a SBS $(Y, \mathcal{C})$ and let f be a positive Borel function on Y s.t. $\forall\ y \in Y$, the sums $\sum_{k=0}^{\infty} f(\sigma^k y)$, $\sum_{k=0}^{\infty} f(\sigma^{-k} y)$ are infinite. Let $X = \{(y, t) \mid 0 \leq t < f(y)\}$. Then, X is the subset of $Y \times \mathbb{R}$ strictly under the graph of f. Give $Y \times \mathbb{R}$ the product Borel structure and restrict it to X. We then obtain a new Borel space $(X, \mathcal{B})$.

A jointly measurable flow $T_t,\ t \in \mathbb{R}$, on X can be defined as follows: a point $(y, u) \in X$ moves vertically up with "unit speed" until it reaches the point $(y, f(y))$ when it goes over to $(\sigma(y), 0)$ and starts moving up again with unit speed. The term "unit speed" means that the linear distance travelled in time t equals t. The point thus reached at time $t > 0$ is defined to be $T_t(y, u)$. For $t < 0$, $T_(y, u)$ is defined to be the point (y', u') s.t. $T_{-,t}(y', u') - (y, u)$. The point $(y, 0)$ is called the *base point of* (y, u).

Analytically, the above is expressible as follows: Let $x = (y, u) \in X$, and let $t \geq 0$. Then, $T_t(x) = T_t(y, u) = \left(\sigma^n y, t + u - \sum_{k=0}^{n-1} f(\sigma^k y)\right)$ where n is the unique integer s.t. $\sum_{k=0}^{n-1} f(\sigma^k y) \leq t + u < \sum_{k=0}^{n} f(\sigma^k y)$. If $t < 0$, the expression is $T_t(x) = \left(\sigma^{-n} y, t + u + \sum_{k=1}^{n} f(\sigma^{-k} y)\right)$ where n is the unique integer s.t. $0 \leq t + u + \sum_{k=1}^{n} f(\sigma^{-k} y) < f(\sigma^{-n} y)$. It is understood that $\sum_{k=0}^{-1}$ and $\sum_{k=0}^{0}$ are equal to zero. It is easy to verify that $T_t,\ t \in \mathbb{R}$ is indeed a flow on X.

The flow $T_t,\ t \in \mathbb{R}$ as defined above is called the *flow (or special flow) built under the function f with base automorphism T and base space* $(Y, \mathcal{C})$. Note that a flow built under a function is a continuous version of automorphism built under a positive integer-valued function. We thus use the notation of T^f for the continuous case also.

Let the base space Y be Polish, the base automorphism σ a homeomorphism of Y and f continuous on Y. Let us give $Y \times \mathbb{R}$ the product topology, where $\mathbb{R}$ has the usual topology. Let $\bar{X} = \{(y, t) \mid 0 \leq t \leq f(y)\}$ be the closure of $X \subseteq Y \times \mathbb{R}$. Now, define $g : \bar{X} \to X$ by $g(y, t) = (y, t)$ if $0 \leq t \leq f(y)$ and $g(y, t) = (\sigma y, o)$ if $t = f(t)$. The map g identifies the point $(f, f(y))$ with $(\sigma y, 0)$. Let Γ be the largest topology on X that makes g continuous. Under this topology, the flow σ^f is a jointly continuous flow of homeomorphisms on X. The topology Γ can be shown to be a Polish Topology.

Therefore, a jointly measurable flow is also jointly continuous wrt a suitable complete separable metric topology, the Polish topology, on X which also generates the σ-algebra $\mathcal{B}$.

Let $T_t,\ t \in \mathbb{R}$ be a jointly measurable flow (without fixed points) on a SBS $(X, \mathcal{B})$. Suppose we are

able to choose on each orbit of T_t a non-empty discrete set of points s.t the totality of these points taken over all orbits is a Borel set in $\mathcal{B}$. That is, we suppose that there exists a Borel set $Y \subseteq X$ s.t. $\forall\ x \in X$, the set $\{t\ :\ T_t(x) \in Y\}$ is a non-empty and discrete subset of $\mathbb{R}$. Such a subset is called a *countable cross-section of the flow*.

Given a countable cross-section Y, we can write X as the union of three Borel sets $I,\ J,\ K$ as: $I = \{x \in X \mid \{t \mid T_t(x) \in Y\}$ is bounded below$\}$, $J = \{x \in X \mid \{t \mid T_t(x) \in Y\}$ is bounded above$\}$, $K = X - I \bigcup J$. Let $i(x) = \inf\{t \mid T_t(x) \in Y\}$ and $j(x) = \sup\{t \mid T_t(x) \in Y\}$. Then i and j are measurable functions, so that I and J, hence, also K, are measurable sets.

Let us write $S(x) = T_{i(x)}(x),\ x \in I$. Then, $S(T_t(x)) = S(x)\ \forall\ t \in \mathbb{R}$ since $i(T_t(x)) = i(x) - t$. The function $S : I \to I$ is again measurable, and constant on orbits. Thus, if we restrict the flow to I then the orbit space admits a Borel cross-section, the image of I under S being the required Borel cross-section. Similarly for J. Therefore, in the set $I \bigcup J$, the flow is isomorphic to a flow built under a function.

Then, there exists a Borel set $Y \subseteq X$ s.t. $\forall\ x \in X$ the set $\{t \mid T_t x \in Y\}$ is non-empty, countable, and discrete in $\mathbb{R}$, the flow is isomorphic to a flow built under a function.

Thus, we note that every jointly measurable flow (without fixed points) on a SBS admits a countable cross-section.

Further, for a jointly measurable T_t, it can be shown [11] that there exists a set $B \in \mathcal{B}$ s.t. $\forall\ x \in X$ the sets $\{t \in \mathbb{R} \mid T_t x \in B\}$ and $\{t \in \mathbb{R} \mid T_t x \notin B\}$ have positive Lebesgue measure.

Then, it can further be shown [11] that every jointly measurable flow $T_t,\ t \in \mathbb{R}$ (without fixed points) on a SBS $(X, \mathcal{B})$ admits a measurable subset $Y \subseteq X$ s.t. $\forall\ x \in X$ the set $\{t \mid T_t x \in Y\}$ is non-empty and discrete in $\mathbb{R}$. Therefore, we see that every jointly measurable flow (without fixed points) on a SBS is isomorphic to a flow built under a function.

For general finite measure preserving flows, this result was proved in [23] while the refinement and adaptation of that method to a descriptive setting can be found in [11].

As a corollary, every jointly measurable flow without fixed points on a SBS $(X, \mathcal{B})$ is a flow of homeomorphisms under a suitable Polish topology on X which generates $\mathcal{B}$.

Furthermore, for a jointly measurable flow $T_t\ t \in \mathbb{R}$ (without fixed points) on a SBS $(X, \mathcal{B})$ and given $0 \le \alpha \le 1$, there exists $B \in \mathcal{B}$ s.t. $\forall\ x \in X$ the orbit of x spends the proportion α of time in B, that is, $\forall\ x \in X$, $\frac{1}{N}$ Lebesgue measure $\{t \mid T_t x \in B,\ 0 \le t < N\} \to \alpha$ as $N \to \infty$.

Note also that, under suitable modifications of the definition of flow built under a function, these results hold for jointly measurable flows with fixed points as well. These will be considered separately at a suitable stage, however, not in the present preparatory paper.

Now, consider the notion of a flow built under a function in a measure theoretic setting. Let $(Y, \mathcal{B}_Y)$ be a SBS equipped with a Borel automorphism $\tau : Y \to Y$ and a σ-finite measure n quasi-invariant for τ. [A measure n on $\mathcal{B}$ is called *quasi-invariant for* τ if $n(B) = 0$ iff $n(\tau B) = 0$ and is called *conservative for* τ if $n(W) = 0$ for every τ-wandering set W.]

Let f be a positive Borel function on Y s.t. $\forall\ y$, the sums $\sum_{k=0}^{\infty} f(\tau^k y)$ and $\sum_{k=1}^{\infty} f(\tau^{-1} y)$ are infinite. Let $T_t,\ t \in \mathbb{R}$ be the flow τ^f built under f with base space $(Y, \mathcal{Y})$ and base automorphism τ. It acts on $Y^f = \{(y,t) \mid 0 \le t < f(y),\ y \in Y\}$.

Let ℓ denote the Lebesgue measure on $\mathbb{R}$ and let the measure n on $Y \times \mathbb{R}$ be restricted to Borel subsets of Y^f. Let us denote this measure on Y^f by $m = m_f$.

The flow $T_t,\ t \in \mathbb{R}$, when considered together with the measure m is called the *flow built under f in a measure theoretic sense*. We call the measure n the *base measure*.

Now, [24], for any $t \in \mathbb{R}$,

$$\frac{dm_t}{dm}(y,u) = \frac{dn_{(t+u)y}}{dn}(y) \quad \text{a.e.m.}$$

where m_t and n_k are the measures $m(T_t(.))$ and $n(\sigma^k(.))$ respectively and $\frac{dm_t}{dm}$ denotes the LRN derivative of a quasi-invariant measure [9].

Recall that the flow $T_t,\ t \in \mathbb{R}$, is the flow σ^f. Then, as a corollary, we also see that m is quasi-invariant under the flow σ^f iff n, the base measure, is quasi-invariant under σ. m is invariant under σ^f iff n is invariant under σ.

Consider now a jointly measurable flow $\tau_t,\ t \in \mathbb{R}$, on a SBS $(X, \mathcal{B})$ equipped with a probability measure m quasi-invariant under the flow. Let us also assume, for simplicity, that the flow $T_t,\ t \in \mathbb{R}$, is free. Then, the map $t \to \tau_t x$ is one-one from $\mathbb{R}$ onto the orbit $\{\tau_t x \mid t \in \mathbb{R}\}$. Thus, a Lebesgue measure is definable on the orbit simply by transferring the Lebesgue measure of $\mathbb{R}$ to it. Let us denote by ℓ_x this Lebesgue measure on the orbit of x under the flow.

Then, $m(A) = 0$ iff $\ell(\{t \mid \tau_t x \in A\}) = \ell_x(A) = 0$ for m-almost every x. [A property which holds for all $x \in X$ except for those x in some m-null set is said to hold m-almost everywhere.]

Let $\tau_t,\ t \in \mathbb{R}$, on $(X, \mathcal{B}, m)$ and $T_t,\ t \in \mathbb{R}$, on $(X', \mathcal{B}', m')$ be two non-singular flows. We say that the two flows are *metrically isomorphic* if there exist

(i) τ_t-invariant m-null set $M \in \mathcal{B}$ and T_t-invariant m'-null set $M' \in \mathcal{B}'$,

(ii) a Borel automorphism ϕ of $X - M$ onto $X' - M'$

s.t. $\forall\, t \in \mathbb{R}$, and $x' \in X' - M'$ we have

(a) $\phi \circ \tau_t \circ \phi^{-1}(x') = T_t x'$,

(b) $m(\phi^{-1}(A')) = 0 \iff m'(A') = 0,\ \forall\, A' \in \mathcal{B}'$,

(c) in case the flows are measure preserving we require $m \circ \phi^{-1} = m'$ in place of above (b).

Then, as shown in [25], every non-singular free flow $\tau_t,\ t \in \mathbb{R}$, on a SPS $(X, \mathcal{B}, m)$ is isomorphic to a flow built under a function in the measure theoretic sense. The function which implements the isomorphism preserves null sets.

On the other hand, the basic theorem of Ambrose [23] states that: every free measure preserving flow on a SPS $(X, \mathcal{B}, m)$ is isomorphic to a flow built under a function in the measure theoretic sense. The function which implements the isomorphism preserves the measure. This holds also if m is a σ-finite measure [26].

VII. NOMENCLATURE ETC.

We have gone to great lengths in reviewing the basics of the ergodic theory because of their relevance to the earlier mentioned significance vis-a-vis P-sets. P-sets are open sets, Borel sets, of the Einstein space $(\mathbb{B}, d)$ and all the above considerations are applicable to P-sets.

The question is, of course, of extending and modifying the relevant of the above point-wise description to suit the P-sets and, needless to say, of extracting physical results from any such mathematical analysis.

To this end, we define some nomenclature that will be used in future relevant works. We note that some of these concepts coincide with already existing mathematical notions.

Firstly, we reserve the word **particle** to refer to a single P-set. We shall refer to any collection of P-sets as an **object** or **objects**. Therefore, an *object* is a *collection of particles*. In the situation that two or more objects unite to become a single object, we shall refer to this as a **merger of objects**. On the other hand, if an object splits into two or more than two objects, we shall refer to this as a **splitting of an object**.

Secondly, if a particle splits into two or more particles, we will call this process a **creation of particles**. On the other hand, two or more particles unite to become a single particle, we will call this process an **annihilation of particles**.

To make mathematically more precise the above ideas and also to conceptually visualize the processes under consideration, the following will be a useful tool.

Consider two non-empty subsets A and B of a topological space (X, Γ). We call the sets A and B *mutually separated* if $A \bigcap B = \emptyset$, *ie*, they are mutually disjoint, and if $A^c \bigcap B = \emptyset$ and $A \bigcap B^c = \emptyset$ where A^c denotes the closure of the set in question. Similarly, we define *mutually touching non-empty sets A and B* as those sets for which $A \bigcap B = \emptyset$ but $A^c \bigcap B \neq \emptyset$ and $A \bigcap B^c \neq \emptyset$. This describes our intuitive notion of touching particles if the sets under consideration are P-sets.

Then, if two touching P-sets *merge* under the action of a Borel automorphism on $\mathbb{B}$, then the boundaries of those P-sets will merge. If a P-set *splits* into two or more P-sets, then the boundary will also split. The above characterization will be useful to describe such processes.

An object is then some suitable collection of such mutually touching sets, *ie*, it is essentially a region bounded by the vanishing of the differential of the volume measure (of eq. (3)) but there are interior points of the region for which the same differential of the volume measure vanishes, so such a region is not a P-set.

Various attributes of a physical particle will be Borel measures on a P-set. Then, using the volume form (of eq. (3)), we may define appropriate quantities averaged over a P-set.

For example, we may define a suitable a.e. finite-valued, positive definite, measurable function, the energy density, on a P-set of $(\mathbb{B}, \mathcal{B}, d)$. When this energy density is integrated wrt the volume measure over a P-set, we may call the resultant quantity as a mass of the physical particle associated with that P-set.

We may associate a Dirac δ-function with this mass and then the "location" of a Dirac δ-particle of this mass will be *intrinsically indeterminate* over the *size* of that P-set.

Such an intrinsic indeterminacy can, thinkably, serve as the origin of the Heisenberg indeterminacy relations in the continuum formulation [30]. That this expectation is indeed true or not will be demonstrated in a later work. Such a demonstration will, obviously, require us to sharpen (mathematically) the notion of *motion* of a particle within the present formulation.

Now, while considering the "motion" of this "particle with mass", we can, of course, have such measures invariant. We will therefore be considering measure preserving automorphisms of the base

space $\mathbb{B}$ (leaving aside the issue of the effect of the automorphism on other P-sets).

Just as we may consider Borel automorphisms of the base space $\mathbb{B}$ that keep a P-set invariant, we may also consider that an object, a collection of P-sets, remains invariant under the automorphism of the base space $\mathbb{B}$.

In the analysis of such situations, we may then consider a *countably finite or infinite chain* of mutually touching non-empty sets as a function $f : \mathbb{N} \to \mathbb{B}$ s.t. $\{A_0, A_1, ...\}$ with A_i and A_{i+1} being mutually touching sets. (The set of natural numbers may be replaced by any directed set here.) Such chains may also remain invariant under the automorphisms of the base space.

At this place, we therefore note that the group of Borel automorphisms of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ is sufficiently large so as to permit such possibilities as above.

VIII. CONCLUDING REMARKS

Now, our ultimate goal is to describe, in precise mathematical terms, different processes of merger of objects, splitting of an object, creation of particles, annihilation of particles in the base or the physical space $\mathbb{B}$.

Furthermore, apart from the above processes, we also want to describe motions of physical objects and other phenomena using the same formalism. This is mainly because the current formalism is supposed to be incorporating the totality of all the possible fields in Nature.

At this point, we then note that in the physical world, we measure distances in terms of a certain *unit of distance*. Distance between two given objects is always an integral multiple of the basic unit of distance. We cannot *measure* any fraction of the basic unit of distance unless, of course, we have a physical object smaller in size than the chosen basic unit of the distance.

For example, let us measure distances in terms of a basic unit - the centimeter. Then, distance between two physical objects is always an integral multiple of a centimeter. Unless we find a physical object smaller than a centimeter, we will be unable to "measure" distances smaller than a centimeter. This is the issue here.

Now, this unit of distance can, of course, be chosen to be a P-set or a suitable collection of P-sets, *ie*, an object. The sizes of P-sets and objects are mathematically well-defined in the present formalism. The metric of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ allows us the precise mathematical description of these conceptions.

Then, given a P-set or an object, we may construct a finite chain of such objects. Then, the distance between two objects can also be measured to be an integral multiple of the "size of a chosen finite chain of objects". This is easily describable within the present formulation.

Clearly, to measure distances smaller than the chosen size of object, we will need another object with "size smaller than that of the first object". Such P-sets and objects always exist is what has emerged in the present formalism.

We have then seen that relevant results from the ergodic theory imply the existence of a suitable Polish topology on the Einstein Borel space $(\mathbb{B}, \mathcal{B})$ whose certain open sets form the class of all P-sets of the metric space $(\mathbb{B}, d)$.

It is then intuitively clear that, using the P-sets or objects, we can always implement the aforementioned construction.

This suggests an appropriate "distance function" on the family of all P-sets/objects of the base space $\mathbb{B}$. Intuitively, it is this *physical distance* that changes with *time parameter* and is the motion in the physical world. This then constitutes the *dynamics in the physical world*.

After all, motion of one body is to be described relative to another body. Of course, we can also consider that two or more bodies do not move relative to each other.

Then, the Borel automorphism of the Einstein space may result into change in the physical distance resulting into relative motion of bodies. On the other hand, an automorphism that keeps invariant a chain of P-sets separating two P-sets can describe the situation of two or more relatively stationary objects.

It is therefore *crucial to distinguish* between the *mathematical metric* (2) and the above mentioned "metric on the class of all P-sets" which we will call a *physical metric*.

We also note here that we can consider countable collections of P-sets and, hence, of objects as defined above. Then, the fact that we can "count objects" in the present formalism also agrees well with our general experience.

Physical objects, each one being an appropriate collection of particles, are "countable" in Nature. For example, we can count the number of chairs, tables, persons in a conference hall. Such physical objects are simply collections of objects. Any theoretical formulation must also be able to describe them as such. This is then the obvious motivation behind our adopting various earlier definitions and nomenclatures which also possess, already existing, mathematical analogues.

The physical metric mentioned above then provides the mathematical notion of the physical dis-

tance between the objects. Effects of the Borel automorphisms of the base space on the physical metric are then to be looked upon as resulting physical motions of physical bodies and various physical phenomena as manifestations of relevant properties of such automorphisms [31].

Essentially, our point of view here is therefore that the *physical dynamics* arises as a result of the Borel automorphisms of the Einstein Borel space $(\mathbb{B}, \mathcal{B})$ and their effects on some suitable physical distance function over the class of P-sets of the Einstein metric space $(\mathbb{B}, d)$.

Then, since we treat physical attributes of a particle as suitable measures, it is also the contention here that the group of Borel automorphisms of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ is sufficiently large to encompass the observed physical phenomena. It is in this broad sense that the present formalism is to be a theory of everything.

Now, one is also struck by the fact that, in the usual formalism of General Relativity, the measuring apparatuses, *e.g.*, rods and clocks, are postulated in a manner that isolates them from the physical phenomena that are being described. For example, the measuring apparatuses in a given spacetime are postulated to be independent of the presence of electromagnetic or other types of fields imposed on that spacetime.

Strictly speaking, measuring apparatuses will have to be treated at par with every other thing that the theory treats, and not be theoretically self-sufficient entities that remain isolated from other physical entities forever.

Then, it is worth pointing out here that the present formalism represents measuring apparatuses in precisely this desired manner. The effects of (Borel) automorphisms of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ on the measuring apparatuses are also accountable in it. Moreover, their treatment [32] is also at par with the treatment of every other thing that the formalism treats.

The present effort constitutes an attempt of developing a theory of the physical universe based on the continuum picture. That it possesses the required simplicity [33] while simultaneously encompassing number of physical phenomena needs to be stressed then. Again, this is due to the fact that the present formalism is to incorporate all the fields of Nature in it.

We then note that the Borel automorphism of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ *cannot* lead to the formation of a *singularity* in any conceivable manner because any such automorphism is a one-to-one and onto map that is also measurable with its inverse also being a measurable map. Consequently, any naked singularities do not arise here as the singularities themselves do not arise.

The question of whether event horizons or black holes can arise within the present formalism is a more complicated one to settle than that of naked singularities.

However, the Borel automorphisms of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ form a group. Then, we can always traverse [34] or cross any given 2-surface both ways. This suggests that a one-way membrane may not arise in the present formalism. It therefore appears that even black holes may not arise in the present formalism.

Above ideas, of physical (as well as mathematical) nature, are mainly intuitive and these ideas will have to be formulated in precise mathematical terms. It is this task that we devote to in a series of papers following this one.

In conclusion, we however note the following. A particle as a concept has existence only at a single spatial location. On the other hand, a continuum has simultaneous existence at more than one, continuous, locations. These are primarily antipodal conceptions.

Newtonian theory is an attempt to understand the happenings of the physical world on the primary basis of the conception of a point-particle. Newton's mechanical world-view codifies all that can be consistently achieved using it.

On the other hand, General Relativity attempts to dispense with this notion of a particle and replaces it with that of an extended particle, the continuum of energy in space.

General Relativity is an attempt to grasp the happenings of the physical world on the basis of the continuum hypothesis. In a definite sense, it is an extension, to include gravity, of the field conception of Faraday and Maxwell in the form of electrodynamics. But, since the laws of electrodynamics of Maxwell and Faraday were linear laws, this extension to gravity required intrinsically non-linear laws. These non-linear laws are the Einstein field equations relating the geometric quantities to the matter quantities.

In the present paper, we argued as to why the continuum description of General Relativity requires the use of a specific spacetime geometry. We then argued as to why we need to restrict to only the 3-dimensional manifold of the Einstein pseudometric (2). This provided us the required notion of extended bodies, bodies as concentrated form of energy in space.

We then showed that the base space of (2) is a Standard Borel Space. The description of the motion of an extended-particle is then permissible along the lines proposed herein.

Whether this description agrees with various experimental or observational results then remains to be investigated and is some issue.

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[1] Einstein A (1970) in *Albert Einstein: Philosopher-Scientist: Autobiographical Notes* (Ed. P. A. Schlipp, La Salle: Open Court Publishing Company - The Library of Living Philosophers, Vol. VII). See, in particular, pages 75, 81, 91 for his views on extending the formalism of General Relativity.

[2] Mach E (1872) *Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit* (Prag: J.G. Calvelsche K. u. K. Universitäts Buchhandlung) - English translation by Philip B. Jourdain (1911): *History and Root of the Principle of the Conservation of Energy* (Chicago: Open Court Publishing Company)
Mach E (1933) *Die Mechanik in ihrer Entwicklung* (Leipzig: F.A. Brockhaus, 9te Auflage) - English translation by Thomas J. McCormack (1960) *The Science of Mechanics* (6th Edition with revisions through the 9th German Edition. LaSalle, Illinois: Open Court Publishing Company)

[3] Wagh S M (2002) *Mach's Principle and Spatial Scale-Invariance of Gravity* **Database: gr-qc/0202005**

[4] Wagh S M (2002) *Classical formulation of Cosmic Censorship Hypothesis* **Database: gr-qc/0201041**

[5] Pais A (1982) *Subtle is the Lord ... The science and the life of Albert Einstein* (Oxford: Clarendon Press)

[6] Joshi K D (1983) *Introduction to General Topology* (Wiley Eastern, New Delhi)

[7] Nadkarni M G (1995) *Basic Ergodic Theory* (Texts and Readings in Mathematics - 6: Hindustan Book Agency, New Delhi)
Cornfeld I P, Fomin S V and Sinai Ya G (1982) *Ergodic Theory* (Springer-Verlag, New York)

[8] Parthasarathy K R (1967) *Probability measures of Metric spaces* (Academic Press, New York)

[9] Halmos P R (1974) *Measure Theory* (Springer-Verlag, New York)
Hewitt E and Stromberg K (1978) *Real and Abstract Analysis* (Springer International Student Edition, Narosa Publishing House, New Delhi)

[10] Weiss B (1984) *Contemporary Mathematics* **26**, 397-421

[11] Wagh V M (1988) *Proc. Indian Acad. (Math. Sci)* **98**, 101-108

[12] Ornstein D S, Rudolph B and Weiss B (1982) *Memoirs of the Amer. Math. Soc.* **37**, No. 262 (end of volume)

[13] Dougherty R, Jackson S, Kechris A (1994) *Trans. Amer. Math. Soc.* **341(4)**, 193-225

[14] Rokhlin V A (1963) *Generators in Ergodic Theory I* (Vestnik Leningrad Univ. Math, 26-32)

[15] Weiss B (1989) *Contemporary Mathematics* **94**, 321-326

[16] Kechris A (1993) *Lectures on Definable Group Actions and Equivalence Relations* (Lecture Notes, Caltech, Pasadena)

[17] Varadrajan V S (1963) *Trans. Amer. Math. Soc.* **109**, 191-220

[18] Hajian A B and Kakutani S (1964) *Trans. Amer. Math. Soc.* **110**, 136-151

[19] Eigen S, Hajian A B, Nadkarni M G (1993) *Proc. Indian Acad. (Math. Sci)* **103**, 321-327

[20] Dye H (1959) *Amer. J. Math.* **81**, 119-159

[21] Chaube P and Nadkarni M G (1989) *Proc. Indian Acad. (Math. Sci)* **99**, 255-261

[22] Krieger W (1970) *On the Araki-Woods Asymptotic Ratio Set and Non-singular Transformations on a Measure Space* (Springer Lecture Notes in Mathematics, No. 160, Springer-Verlag, New York), 158-177 and references therein

[23] Ambrose W (1941) *Annals of Mathematics* **42**, 723-739

[24] Mathew J and Nadkarni M G (1982) in *Statistics and Probability: Essays in honor of C R Rao* (edited by G Kallianpur, P Krishnaih, J K Ghosh, North-Holland, 493-504)

[25] Dani S G (1976) *Amer. J. Math.* **98**, 119-163 and references therein

[26] Krengel U (1968) *Math. Ann.* **176**, 191-190; **182**, 1-39.

[27] As Einstein [1] says it: "The right side (of $R_{ik} - \frac{1}{2}g_{ik}R = -k\,T_{ik}$: my brackets) is a formal condensation of all things whose comprehension in the sense of a field-theory is still problematic." He is referring to the comprehension of the energy-momentum tensor in terms of "extended" bodies (fields) and not point-particles. That he recognizes the dependence of the usual conception of the energy-momentum tensor on that of a point-particle is clear.

[28] Girish Sahasrabudhe suggested this phraseology

[29] This is, in a definite sense, the field-theoretic comprehension of the energy-momentum tensor.

[30] Note that Einstein [1] regarded the correctness of Heisenberg's indeterminacy relations as being "finally demonstrated". However, as is well known, he differed from Bohr [1] and others on the issue of the "origin" of the indeterminacy relations.

[31] Therefore, a joint manifestation of Borel automorphisms of the Einstein Borel space $(\mathbb{B}, \mathcal{B})$ and the association of a Dirac δ-particle with a P-set is behind Heisenberg's indeterminacy relations in the present continuum formulation.

[32] A measuring rod is a suitable chain of P-sets or objects. A measuring clock is a P-set or an object undergoing periodic motion. That their treatment in the present formalism is the same as that of everything else is self-evident.
The comprehension of such obvious "measuring apparatuses" will require us to work out precisely the corresponding Borel automorphisms and their effects on the P-sets or objects under consideration. This type of treatment will be presented in a later work.

[33] Recall here a statement by Einstein that [1]: A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability.

[34] Let a Borel automorphism of the Einstein space $(\mathbb{B}, \mathcal{B}, d)$ result into a entry into the given 2-surface from one direction. The inverse of that Borel automorphism (resulting into entry in the other direction out of that 2-surface) exists to reverse the first entry is the point here.

GLOSSARY OF NOTATIONS

SBS	Standard Borel Space
SMS	Standard Measure Space
SPS	Standard Probability Space
wrt	with respect to
s.t.	such that
a.e.	almost everywhere
μ-a.e.	modulo a set of μ-measure zero
fimp	for infinitely many positive
fimn	for infinitely many negative
$s_T(A)$	$= \bigcup_{k=-\infty}^{\infty} T^k A$: *T-saturation of A*
B-shift	Bernoulli shift
K-shift	Kolmogorov shift
DAM	Diadic Adding Machine or Odometer
$\mathfrak{P}(X)$	Power set of a given set X
$A \triangle B$	Symmetric union $= (A - B) \bigcup (B - A)$
$\mathcal{B}$ or $\mathcal{B}_X$	Borel σ-algebra of X
$\mathcal{N}$	σ-ideal of subsets of a given set X
$A \sim B$	equivalence of sets A and B by countable decomposition
$A \sim B \,(\mathrm{mod}\, \mathcal{N})$	equivalence of sets A and B by countable decomposition $(\mathrm{mod}\, \mathcal{N})$
$\mathcal{W}_T$	σ-ideal generated by all T-wandering sets in $\mathcal{B}$ or the Shelah-Weiss ideal
$\mathcal{H}$	the Hopf ideal
$A \prec\prec B$	if $\exists$ a measurable subset $C \subseteq B$ s.t. $A \sim C$ and $s_T(B - C) = s_T B$
$A \prec\prec B \,(\mathrm{mod}\, \mathcal{N})$	if $\exists$ a set $N \in \mathcal{N}$ s.t. $A \triangle N \prec\prec B \triangle N$
A is *T-compressible*	if $A \prec\prec A$
$T_1 \prec T_2$	if T_1 is isomorphic to $(T_1)_A$ for some $A \in \mathcal{B}$ with $\bigcup_{k=0}^{\infty} T_1^k A = X$, that is, if automorphism T_1 is a *derivative* of T_2 or if T_2 is a *primitive* of T_1
$T_1 \underset{\mathrm{K}}{\sim} T_2$	Kakutani equivalent automorphisms